LOGARITHM LAWS FOR FLOWS ON HOMOGENEOUS SPACES

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Abstract. In this paper we generalize and sharpen D. Sullivan's logarithm law for geodesics by specifying conditions on a sequence of subsets $\{A_t \mid t \in \mathbb{N}\}$ of a homogeneous space G/Γ (G a semisimple Lie group, Γ an irreducible lattice) and a sequence of elements f_t of G under which $\#\{t \in \mathbb{N} \mid f_t x \in A_t\}$ is infinite for a.e. $x \in G/\Gamma$. The main tool is exponential decay of correlation coefficients of smooth functions on G/Γ. Besides the general (higher rank) version of Sullivan's result, as a consequence we obtain a new proof of the classical Khinchin-Groshev theorem on simultaneous Diophantine approximation, and settle a conjecture recently made by M. Skriganov.

§1. Introduction

1.1. This work has been motivated by the following two related results. The first one is the Khinchin-Groshev theorem, one of the cornerstones of metric theory of Diophantine approximation. We will denote by $M_{m,n}(\mathbb{R})$ the space of real matrices with m rows and n columns, and by $\|\cdot\|$ the norm on $\mathbb{R}^k$, $k \in \mathbb{N}$, given by $\|\mathbf{x}\| = \max_{1 \le i \le k} |x_i|$.

Theorem [G]. *Let m, n be positive integers and $\psi : [1, \infty) \mapsto (0, \infty)$ a non-increasing continuous function. Then for almost every (resp. almost no) $A \in M_{m,n}(\mathbb{R})$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\|A\mathbf{q} + \mathbf{p}\|^m \le \psi(\|\mathbf{q}\|^n) \quad \textit{for some } \mathbf{p} \in \mathbb{Z}^m\,, \tag{1.1}$$

provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).

1.2. The second motivation comes from the paper [Su] of D. Sullivan. Let $\mathbb{H}^{k+1}$ stand for the $k+1$-dimensional real hyperbolic space with curvature -1. Take a discrete group Γ of hyperbolic isometries of $\mathbb{H}^{k+1}$ such that $Y = \mathbb{H}^{k+1}/\Gamma$ is not compact and has finite volume. For $y \in Y$, denote by $S_y(Y)$ the set of unit vectors tangent to Y at y, and by $S(Y)$ the unit tangent bundle $\{(y,\xi) \mid y \in Y,\, \xi \in S_y(Y)\}$ of Y. Finally, for $(y,\xi) \in S(Y)$ let $\gamma_t(y,\xi)$ be the geodesic on Y through y in the direction of ξ. The following theorem is essentially proved in [Su] (see Remark (1) in §9):

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Theorem. *For Y as above, fix $y_0 \in Y$, and let $\{r_t \mid t \in \mathbb{N}\}$ be an arbitrary sequence of real numbers. Then for any $y \in Y$ and almost every (resp. almost no) $\xi \in S_y(Y)$ there are infinitely many $t \in \mathbb{N}$ such that*

$$\operatorname{dist}\big(y_0, \gamma_t(y,\xi)\big) \geq r_t\,, \tag{1.2}$$

provided the series $\sum_{t=1}^{\infty} e^{-kr_t}$ diverges (resp. converges).

1.3. A choice $r_t = \frac{1}{\varkappa}\log t$, where $\varkappa$ is arbitrarily close to k, yields the following statement, which has been referred to as the *logarithm law for geodesics*:

Corollary. *For Y as above, any $y \in Y$ and almost all $\xi \in S_y(Y)$,*

$$\limsup_{t\to\infty} \frac{\operatorname{dist}\big(y, \gamma_t(y,\xi)\big)}{\log t} = 1/k\,. \tag{1.3}$$

1.4. It seems natural to ask whether one can generalize the statements of Theorem 1.2 and Corollary 1.3 to other locally symmetric spaces of noncompact type. On the other hand, Sullivan used a geometric proof of the case $m = n = 1$ of Theorem 1.1 to prove Theorem 1.2; thus one can ask whether there exists a connection between the general case of the Khinchin-Groshev theorem and some higher rank analogue of Sullivan's result.

In this paper we answer both questions in the affirmative. In particular, the following generalization of Sullivan's results can be proved:

Theorem. *For any noncompact irreducible*[1] *locally symmetric space Y of noncompact type and finite volume there exists $k = k(Y) > 0$ such that the following holds: if $y_0 \in Y$ and $\{r_t \mid t \in \mathbb{N}\}$ is an arbitrary sequence of positive numbers, then for any $y \in Y$ and almost every (resp. almost no) $\xi \in S_y(Y)$ there are infinitely many $t \in \mathbb{N}$ such that* (1.2) *is satisfied, provided the series $\sum_{t=1}^{\infty} e^{-kr_t}$ diverges (resp. converges). Consequently,* (1.3) *holds for any $y \in Y$ and almost all $\xi \in S_y(Y)$.*

The constant $k(Y)$ can be explicitly calculated in any given special case; in fact, $k(Y) = \lim_{r\to\infty} -\log\big(\operatorname{vol}(A(r))\big)/r$, where

$$A(r) \overset{\text{def}}{=} \{y \in Y \mid \operatorname{dist}(y_0, y) \geq r\}\,, \tag{1.4}$$

and "vol" stands for a Riemannian volume. In other words, the series $\sum_{t=1}^{\infty} e^{-kr_t}$ is, up to a constant, the sum of volumes of sets $A(r_t)$. The latter sets can be viewed as a "target shrinking to ∞" (cf. [HV]), and Theorems 1.2 and 1.4 say that if the shrinking is slow enough (read: the sum of the volumes is infinite), then almost all geodesics approach infinity faster than the sets $A(r_t)$.

This "shrinking target" phenomenon, being one of the main themes of the present paper, deserves an additional discussion. Thus we have to make a terminological digression. Let (X, μ) be a probability space and let $F = \{f_t \mid t \in \mathbb{N}\}$ be a sequence of μ-preserving transformations of X. Also let $\mathcal{B}$ be a family of measurable subsets of X.

[1]In fact the theorem is true for reducible spaces as well, see §10.2 for details.

1.5. Definition. Say that $\mathcal{B}$ is a *Borel-Cantelli family for F* if for every sequence $\{A_t \mid t \in \mathbb{N}\}$ of sets from $\mathcal{B}$ one has

$$\mu\big(\{x \in X \mid f_t(x) \in A_t \text{ for infinitely many } t \in \mathbb{N}\}\big) = \begin{cases} 0 & \text{if} \quad \sum_{t=1}^{\infty} \mu(A_t) < \infty \\ 1 & \text{if} \quad \sum_{t=1}^{\infty} \mu(A_t) = \infty \end{cases}$$

Note that the statement on top is always true in view of the classical Borel-Cantelli Lemma, see §2.3. An important special case is $F = \{f^t \mid t \in \mathbb{N}\}$ for a measure-preserving transformation $f : X \mapsto X$. We will say that $\mathcal{B}$ is *Borel-Cantelli for f* if it is Borel-Cantelli for F as above.

It is easy to see that $f : X \mapsto X$ is ergodic (resp. weakly mixing[2]) iff every one-element (resp. finite) family of sets of positive measure is Borel-Cantelli for f. On the other hand, if (X, μ) is nontrivial, then for any sequence of transformations $F = \{f_t\}$ one can construct a family (say, $A_t = f_t(A)$ with $0 < \mu(A) < 1$) which is not Borel-Cantelli for F. Therefore in order to describe Borel-Cantelli families of sets for a particular sequence of maps, it is natural to specialize and impose certain regularity restrictions on the sets considered.

An important example is given in the paper [P] of W. Philipp: there $X = [0, 1]$, f is an expanding map of X given by either $x \mapsto \{\theta x\}$, $\theta > 1$, or by $x \mapsto \{\frac{1}{x}\}$ ($\{\cdot\}$ stands for the fractional part), and it is proved that the family of all intervals is Borel-Cantelli for f. This means that one can take any $x_0 \in [0, 1]$ and consider a "target shrinking to x_0", i.e. a sequence $(x_0 - r_t, x_0 + r_t)$. Then almost all orbits $\{f^t x\}$ get into infinitely many such intervals whenever r_t decays slowly enough. This can be thought of as a quantitative strengthening of density of almost all orbits (cf. the paper [Bos] for a similar approach to the rate of recurrence).

We postpone further discussion of this general set-up until §10.2, and concentrate on "targets shrinking to infinity" in noncompact spaces. Our goal is to state a result which will imply both Theorem 1.4 and Theorem 1.1. For Y as in Theorem 1.4, let G be the connected component of the identity in the isometry group of the universal cover of Y. Then G is a connected semisimple Lie group without compact factors, and the space Y can be identified with $K\backslash G/\Gamma$, where Γ is an irreducible lattice in G and K is a maximal compact subgroup of G. Instead of working with Y, we choose the homogeneous space $X = G/\Gamma$ as our main object of investigation. Fix a Cartan subalgebra $\mathfrak{a}$ of the Lie algebra of G. It is known [Ma] that the geodesic flow on the unit tangent bundle $S(Y)$ of Y can be realized via action of one-parameter subgroups of the form $\{\exp(t\mathbf{z})\}$, with $\mathbf{z} \in \mathfrak{a}$, on the space X (see §6 for details). In what follows, we will choose a maximal compact subgroup K of G, endow X with a Riemannian metric by fixing a right invariant Riemannian metric on G bi-invariant with respect to K, and let μ be the normalized Haar measure on X.

Recall that the "neighborhoods of ∞" of Theorem 1.4 are the complements $A(r)$, see (1.4), of balls in Y, and it follows from that theorem that the family $\{A(r) \mid r > 0\}$ is Borel-Cantelli for the time-one map of the geodesic flow. To describe sequences of sets "shrinking to infinity" in X, we will replace the distance function $\operatorname{dist}(y_0, \cdot)$ by a function Δ on X satisfying certain properties, and consider the family

$$\mathcal{B}(\Delta) \stackrel{\text{def}}{=} \big\{\{x \in X \mid \Delta(x) \geq r\} \mid r \in \mathbb{R}\big\}$$

of super-level sets of Δ. To specify the class of functions Δ that we will work with, we introduce the following

[2]This characterization of weak mixing was pointed out to us by Y. Guivarc'h and A. Raugi; see also [CK].

1.6. Definition. For a function Δ on X, define the *tail distribution function* Φ_Δ of Δ by

$$\Phi_\Delta(z) \stackrel{\text{def}}{=} \mu\big(\{x \mid \Delta(x) \ge z\}\big)\,.$$

Now say that Δ is *DL* (an abbreviation for "distance-like") if it is uniformly continuous, and Φ_Δ does not decrease very fast, more precisely, if

$$\text{(DL)} \qquad \exists\, c, \delta > 0 \text{ such that } \Phi_\Delta(z+\delta) \ge c \cdot \Phi_\Delta(z) \ \forall z \ge 0\,.$$

For $k > 0$, we will also say that Δ is *k-DL* if it is uniformly continuous and in addition

$$(k\text{-DL}) \qquad \exists\, C_1, C_2 > 0 \text{ such that } C_1 e^{-kz} \le \Phi_\Delta(z) \le C_2 e^{-kz} \quad \forall\, z \in \mathbb{R}\,.$$

It is clear that (k-DL) implies (DL). Note that DL functions on X exist only when X is not compact (see §4.3). The most important example (§5) is the distance function on X. Thus the following theorem can be viewed as a generalization of Theorem 1.4:

1.7. Theorem. *Let G be a connected semisimple Lie group without compact factors, Γ an irreducible lattice in G, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G, $\mathbf{z}$ a nonzero element of $\mathfrak{a}$. Then:*

(a) *if Δ is a DL function on $X = G/\Gamma$, the family $\mathcal{B}(\Delta)$ is Borel-Cantelli for* $\exp(\mathbf{z})$*;*

(b) *if Δ is k-DL for some $k > 0$, then for almost all $x \in X$ one has*

$$\limsup_{t\to+\infty} \frac{\Delta\big(\exp(t\mathbf{z})x\big)}{\log t} = 1/k\,. \tag{1.5}$$

In particular, (1.3) can be derived from (1.5) by taking $G = SO_{k+1,1}(\mathbb{R})$ and $\Delta(x) = \operatorname{dist}(x_0, x)$ for fixed $x_0 \in G/\Gamma$.

1.8. In fact, it is possible to derive a version of Theorem 1.7 for actions of multi-parameter subgroups of G. More generally, we will consider actions of arbitrary countable sequences $\{f_t \mid t \in \mathbb{N}\}$ of elements of G. To specify a class of sequences good for our purposes, denote by $\|g\|$ the distance between $g \in G$ and the identity element of G, and say that a sequence $\{f_t\}$ is *ED* (an abbreviation for "exponentially divergent") if

$$\text{(ED)} \qquad \sup_{t\in\mathbb{N}} \sum_{s=1}^{\infty} e^{-\beta\|f_s f_t^{-1}\|} < \infty \quad \forall\, \beta > 0\,.$$

In this setting we state the following general result:

Theorem. *For G and Γ as in Theorem 1.7, let $F = \{f_t \mid t \in \mathbb{N}\}$ be an ED sequence of elements of G and Δ a DL function on G/Γ. Then the family $\mathcal{B}(\Delta)$ is Borel-Cantelli for F.*

1.9. Clearly Theorem 1.7 is a special case of the above theorem: it is easy to check (see §4.4) that the sequence $f_t = \exp(t\mathbf{z})$, with $\mathbf{z} \in \mathfrak{a} \smallsetminus \{0\}$, satisfies (ED). More generally, the following multi-parameter generalization of Theorem 1.7 can be derived from Theorem 1.8:

Theorem. *For G, Γ, X and $\mathfrak{a}$ as in Theorem 1.7,*

(a) *if Δ is a DL function on X, and $t \mapsto \mathbf{z}_t$ is a map from $\mathbb{N}$ to $\mathfrak{a}$ such that*

$$\inf_{t_1\neq t_2} \|\mathbf{z}_{t_1} - \mathbf{z}_{t_2}\| > 0\,, \tag{1.6}$$

then the family $\mathcal{B}(\Delta)$ is Borel-Cantelli for $\{\exp(\mathbf{z}_t) \mid t \in \mathbb{N}\}$;

(b) *if Δ is k-DL for some $k > 0$, and $\mathfrak{d}_+$ is a nonempty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$ ($1 \le d \le \operatorname{rank}_{\mathbb{R}}(G)$), then for almost all $x \in X$ one has*

$$\limsup_{\mathbf{z}\in\mathfrak{d}_+,\, \mathbf{z}\to\infty} \frac{\Delta\big(\exp(\mathbf{z})x\big)}{\log\|\mathbf{z}\|} = d/k\,. \tag{1.7}$$

1.10. From the above theorem one can get logarithm laws for flats in locally symmetric spaces. Let the space Y be as in Theorem 1.4. As usual, by a d-dimensional flat in Y $(1 \le d \le \operatorname{rank}(Y))$ we mean the image of $\mathbb{R}^d$ under a locally isometric embedding into Y. For $y \in Y$, denote by $S^d_y(Y)$ the set of orthonormal d-tuples of vectors $\xi_i \in S_y(Y)$ which form a basis for a tangent space to a flat passing through y. The set $S^d_y(Y)$ is a real algebraic variety coming with the natural measure class, which makes it possible to talk about "almost all flats passing through y". If $\vec{\xi} = (\xi_1, \dots, \xi_d) \in S^d_y(Y)$, we will denote by $\mathbf{t} = (t_1, \dots, t_d) \mapsto \gamma_{\mathbf{t}}(y, \vec{\xi})$ the embedding specified by $\vec{\xi}$, that is, we let $\gamma_{\mathbf{t}}(y, \vec{\xi}) \stackrel{\text{def}}{=} \exp_y(\sum_i t_i \xi_i)$ (a multi-dimensional analog of the geodesic in the direction of a single vector $\xi \in S_y(Y)$).

Theorem. *Let Y, y_0 and $k = k(Y)$ be as in Theorem 1.4. Take $1 \le d \le \operatorname{rank}(Y)$ and a nonempty open cone $\mathfrak{d}_+ \subset \mathbb{R}^d$, and let $\mathbf{t} \mapsto r_{\mathbf{t}}$, $\mathbf{t} \in \mathfrak{d}_+ \cap \mathbb{Z}^d$, be a real-valued function. Then for any $y \in Y$ and almost every (resp. almost no) $\vec{\xi} \in S^d_y(Y)$ there are infinitely many $\mathbf{t} \in \mathfrak{d}_+ \cap \mathbb{Z}^d$ such that $\operatorname{dist}\big(y_0, \gamma_{\mathbf{t}}(y, \vec{\xi})\big) \ge r_{\mathbf{t}}$, provided the series $\sum_{\mathbf{t} \in \mathfrak{d}_+ \cap \mathbb{Z}^d} e^{-kr_{\mathbf{t}}}$ diverges (resp. converges). Consequently, for any $y \in Y$ and almost all $\vec{\xi} \in S^d_y(Y)$ one has*

$$\limsup_{\mathbf{t} \in \mathfrak{d}_+,\, \mathbf{t} \to \infty} \frac{\operatorname{dist}\big(y, \gamma_{\mathbf{t}}(y, \vec{\xi})\big)}{\log \|\mathbf{t}\|} = d/k\,. \tag{1.8}$$

1.11. Another class of applications of Theorems 1.7 and 1.9 is given by a modification of S.G. Dani's [D, §2] correspondence between Diophantine approximation of systems of m linear forms in n variables and flows on the space of lattices in $\mathbb{R}^k$, where $k = m + n$. Namely, consider $G = SL_k(\mathbb{R})$, $\Gamma = SL_k(\mathbb{Z})$, and the function Δ on the space G/Γ of unimodular lattices in $\mathbb{R}^k$ defined by

$$\Delta(\Lambda) \stackrel{\text{def}}{=} \max_{\mathbf{v} \in \Lambda \smallsetminus \{0\}} \log\left(\tfrac{1}{\|\mathbf{v}\|}\right). \tag{1.9}$$

Denote also by f_t the element of G of the form

$$f_t = \operatorname{diag}(\underbrace{e^{t/m}, \dots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \dots, e^{-t/n}}_{n \text{ times}})\,. \tag{1.10}$$

We will show in §8 that Theorem 1.1 follows from the fact that the family $\mathcal{B}(\Delta)$ is Borel-Cantelli for f_1. Using similar technique, one can also prove a result that was, in somewhat weaker form, conjectured by M. Skriganov in [Sk]:

Theorem. *Let $\psi : [1, \infty) \mapsto (0, \infty)$ be a non-increasing continuous function and k an integer greater than 1. Then for almost every (resp. almost no) unimodular lattice Λ in $\mathbb{R}^k$ there are infinitely many $\mathbf{v} \in \Lambda$ such that*

$$\Pi(\mathbf{v}) \le \|\mathbf{v}\| \cdot \psi(\|\mathbf{v}\|) \tag{1.11}$$

(here and hereafter we use the notation $\Pi(\mathbf{v}) \stackrel{\text{def}}{=} \prod_{i=1}^k |v_i|$ for $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{R}^k$), provided the integral $\int_1^\infty (\log x)^{k-2} \psi(x)\, dx$ diverges (resp. converges).

In §9 we will explain why the above statement can be thought of as a higher-dimensional multiplicative generalization of Khinchin's Theorem, and how one can derive it from Theorem 1.9 by considering the action of the whole Cartan subgroup of $SL_k(\mathbb{R})$ on the space $SL_k(\mathbb{R})/SL_k(\mathbb{Z})$.

The paper is organized as follows. In §2 we work in a general setting of a probability space (X, μ) and a sequence of nonnegative measurable functions $\mathcal{H} = \{h_t \mid t \in \mathbb{N}\}$ on X, and, following V. Sprindžuk, write down a condition (Lemma 2.6) which guarantees that for almost every $x \in X$ the sum $\sum_{t=1}^{\infty} h_t(x)$ is infinite. Then we throw in a measure preserving action of $F = \{f_t \mid t \in \mathbb{N}\}$ and apply the aforementioned results to the *twisted sequence* $\mathcal{H}^F \stackrel{\text{def}}{=} \{f_t^{-1} h_t\}$.

In §3 we restrict ourselves to flows on G/Γ and prove the following

1.12. Theorem. *Let G be a connected semisimple center-free Lie group without compact factors, Γ an irreducible lattice in G, and let ρ_0 stand for the regular representation of G on the subspace of $L^2(G/\Gamma)$ orthogonal to constant functions. Assume in addition that G/Γ is not compact. Then the restriction of ρ_0 to any simple factor of G is isolated (in the Fell topology) from the trivial representation.*

The latter condition is known (cf. [KM, §2.4]) to guarantee exponential decay of correlation coefficients of smooth functions on G/Γ, see Corollary 3.5. In the next section we use the fact that Δ is DL to approximate characteristic functions of the sets $\{x \in G/\Gamma \mid \Delta(x) \geq r_t\}$ by smooth functions h_t. A quantitative strengthening of Theorem 1.8 is then proved by deriving Sprindžuk's condition from estimates on decay of correlation coefficients of functions h_t. Theorem 1.9 (hence 1.7 as well) is also proved in §4. After that we describe applications to geodesics and flats in locally symmetric spaces (Theorems 1.4 and 1.10) and Diophantine approximation (Theorems 1.1 and 1.11).

§2. Borel-Cantelli-type results

2.1. Let (X, μ) be a probability space. We will use notation $\mu(h) = \int_X h\, d\mu$ for an integrable function h on X. Let us consider sequences $\mathcal{H} = \{h_t \mid t \in \mathbb{N}\}$ of nonnegative integrable[3] functions on X, and, for $N = 1, \dots, \infty$, denote

$$S_{\mathcal{H},N}(x) \stackrel{\text{def}}{=} \sum_{t=1}^{N} h_t(x) \quad \text{and} \quad E_{\mathcal{H},N} \stackrel{\text{def}}{=} \sum_{t=1}^{N} \mu(h_t) = \mu(S_{\mathcal{H},N})\,;$$

this notation will be used throughout the paper. We will omit the index $\mathcal{H}$ when it is clear from the context. A special case of such a sequence is given by characteristic functions $h_t = 1_{A_t}$, where $\mathcal{A} = \{A_t \mid t \in \mathbb{N}\}$ is a sequence of measurable subsets of X. In this case we will put the index $\mathcal{A}$ in place of $\mathcal{H}$ in the above notation. We will say that a sequence $\mathcal{H}$ (resp. $\mathcal{A}$) of functions (resp. sets) is *summable* if $E_{\mathcal{H},\infty}$ (resp. $E_{\mathcal{A},\infty}$) is finite, and *nonsummable* otherwise.

Main example. If Δ is any function on X and $\{r_t \mid t \in \mathbb{N}\}$ a sequence of real numbers, one can consider the sequence of super-level sets $\{x \mid \Delta(x) \geq r_t\}$ of Δ; their measures are equal to $\Phi_\Delta(r_t)$, where Φ_Δ is the tail distribution function (see §1.6) of Δ.

2.2. Another main example. Let $F = \{f_t \mid t \in \mathbb{N}\}$ be a sequence of μ-preserving transformations of X. Then given any sequence $\mathcal{H} = \{h_t \mid t \in \mathbb{N}\}$ of functions on X or a sequence $\mathcal{A} = \{A_t \mid t \in \mathbb{N}\}$ of subsets of X, one can consider *twisted* sequences

$$\mathcal{H}^F \stackrel{\text{def}}{=} \{f_t^{-1} h_t \mid t \in \mathbb{N}\} \quad \text{and} \quad \mathcal{A}^F \stackrel{\text{def}}{=} \{f_t^{-1} A_t \mid t \in \mathbb{N}\}\,.$$

By F-invariance of μ, $E_{\mathcal{H}^F,N}$ is the same as $E_{\mathcal{H},N}$ for any $N \in \mathbb{N}$; in particular, the twisted sequence is summable if and only if the original one is.

[3] Throughout the sequel all the functions h_t will be assumed measurable, integrable, a.e. nonnegative and nonzero on a set of positive measure.

2.3. Given a sequence $\mathcal{A} = \{A_t \mid t \in \mathbb{N}\}$ and a μ-generic point $x \in X$, one may want to look at the asymptotics of $S_{\mathcal{A},N}(x) = \#\{1 \le t \le N \mid x \in A_t\}$ in comparison with the sum $E_{\mathcal{A},N}$ of measures of the sets A_t, $1 \le t \le N$, as $N \to \infty$. This is for example the subject of the classical Borel-Cantelli Lemma. In general, for a sequence $\mathcal{H}$ of functions on X, it is very easy to estimate the ratio of $S_{\mathcal{H},N}(x)$ and $E_{\mathcal{H},N}$ from above as follows:

Lemma (cf. [KS, part (i) of the Theorem]). *Let (X, μ) be a probability space, $\mathcal{H}$ a sequence of functions on X. Then*

$$\liminf_{N\to\infty} \frac{S_{\mathcal{H},N}(x)}{E_{\mathcal{H},N}} < \infty \quad \text{for } \mu\text{-a.e. } x \in X\,.$$

In particular, if $\mathcal{H}$ is summable, $S_{\mathcal{H},\infty}$ is finite almost everywhere.

Proof. By the Fatou Lemma, $\mu\Big(\liminf_{N\to\infty} \frac{S_{\mathcal{H},N}}{E_{\mathcal{H},N}}\Big) \le \liminf_{N\to\infty} \mu\big(\frac{S_{\mathcal{H},N}}{E_{\mathcal{H},N}}\big) = 1.$ □

One immediately recognizes the last assertion as the conclusion of the easy part of the classical Borel-Cantelli Lemma. It takes care of the convergence cases in all the Khinchin-type theorems stated in the introduction, as well as of the upper estimates for the limits in logarithm laws (1.3), (1.5), (1.7) and (1.8).

2.4. The corollary below will make the connection with logarithm laws more transparent. We need the following terminology: a real-valued function $r(\cdot)$ will be called *quasi-increasing* if there exists a constant C such that

$$r(t_2) > r(t_1) - C \text{ whenever } t_1 \le t_2 < t_1 + 1\,. \tag{2.1}$$

Corollary. *Let X be a metric space, μ a probability measure on X, $d, k \in \mathbb{N}$, $\mathfrak{d}_+ \subset \mathbb{R}^d$ a nonempty open cone, $\mathbf{z} \mapsto f_\mathbf{z}$ a continuous*[4] *homomorphism from $\mathfrak{d}_+$ to the semigroup of all self-maps of X, Δ a k-DL function on X. For some $t_0 \in \mathbb{R}$, let $r : [t_0, \infty) \mapsto \mathbb{R}$ be a quasi-increasing function such that the integral*

$$\int_{t_0}^{\infty} t^{d-1} e^{-kr(t)}\, dt \tag{2.2}$$

converges. Then for μ-almost all $x \in X$ one has $\Delta\big(f_\mathbf{z}(x)\big) < r(\|\mathbf{z}\|)$ whenever $\mathbf{z} \in \mathfrak{d}_+$ is far enough from 0. Consequently,

$$\limsup_{\mathbf{z}\in\mathfrak{d}_+,\, \mathbf{z}\to\infty} \frac{\Delta\big(f_\mathbf{z}(x)\big)}{\log\|\mathbf{z}\|} \underset{\mu\text{-a.e.}}{\le} d/k\,. \tag{2.3}$$

Proof. Choose a lattice Σ in $\mathbb{R}^d$; from (2.1) and the convergence of (2.2) it follows that the series

$$\sum_{\mathbf{z}\in\mathfrak{d}_+\cap\Sigma,\, \|\mathbf{z}\|\ge t_0} e^{-kr(\|\mathbf{z}\|)} \tag{2.4}$$

converges. Clearly for any $\mathbf{z} \in \mathfrak{d}_+$ far enough from 0 one can find $\mathbf{z}' \in \mathfrak{d}_+ \cap \Sigma$ such that

$$\|\mathbf{z}\| - 1 \le \|\mathbf{z}'\| \le \|\mathbf{z}\|\,, \tag{2.5}$$

[4] Here by the distance between two maps $f_1, f_2 : X \mapsto X$ we mean $\sup_{x\in X} \operatorname{dist}_X\big(f_1(x) - f_2(x)\big)$.

and $\|\mathbf{z}' - \mathbf{z}\|$ is less than some uniform constant C_1. Since the correspondence $\mathbf{z} \mapsto f_{\mathbf{z}}$ is continuous, for some C_2 one then has $\sup_{x\in X} \operatorname{dist}\big(f_{\mathbf{z}}(x), f_{\mathbf{z}'}(x)\big) < C_2$; further, from the uniform continuity of Δ it follows that for some C_3 one has

$$\sup_{x\in X} |\Delta\big(f_{\mathbf{z}}(x)\big) - \Delta\big(f_{\mathbf{z}'}(x)\big)| < C_3\,. \tag{2.6}$$

Now consider the sequence of sets $\mathcal{A} \stackrel{\text{def}}{=} \big\{\{x \in X \mid \Delta(x) \geq r(\|\mathbf{z}\|) - C - C_3\} \bigm| \mathbf{z} \in \mathfrak{d}_+ \cap \Sigma\big\}$, with C as in (2.1), and $F = \{f_{\mathbf{z}} \mid \mathbf{z} \in \mathfrak{d}_+ \cap \Sigma\}$. It follows from the convergence of (2.4) and Δ being DL that $\mathcal{A}$ is summable. Applying Lemma 2.3 to the twisted sequence $\mathcal{A}^F$, one concludes that for almost all x one has $\Delta\big(f_{\mathbf{z}'}(x)\big) < r(\|\mathbf{z}'\|) - C - C_3$ for $\mathbf{z}' \in \mathfrak{d}_+ \cap \Sigma$ with large enough $\|\mathbf{z}'\|$. In view of (2.1), (2.5) and (2.6), this implies that for almost all x one has $\Delta\big(f_{\mathbf{z}}(x)\big) < r(\|\mathbf{z}\|)$ for all $\mathbf{z} \in \mathfrak{d}_+$ with large enough $\|\mathbf{z}\|$. The second part of the corollary is obtained by taking $r(t) = \frac{d}{\varkappa}\log t$ with $\varkappa < k$. The integral (2.2) obviously converges, therefore for almost all x one has $\dfrac{\Delta\big(f_{\mathbf{z}}(x)\big)}{\log\|\mathbf{z}\|} < \dfrac{d}{\varkappa}$ whenever $\mathbf{z} \in \mathfrak{d}_+$ is far enough from 0, and (2.3) follows. $\square$

2.5. Example. Take $X = S(Y)$ as in §1.2, μ the Liouville measure on $S(Y)$, fix $y_0 \in Y$ and let $\Delta\big((y,\xi)\big) = \operatorname{dist}(y_0, y)$. As mentioned in [Su, §9], Δ is k-DL. From the above corollary (with $d = 1$ and $\mathfrak{d}_+ = \mathbb{R}_+$) one concludes that $\limsup_{t\to\infty} \dfrac{\operatorname{dist}\big(y_0, \gamma_t(y,\xi)\big)}{\log t}$ as $t \to \infty$ is not greater than $1/k$. To derive the upper estimate for the limit in Corollary 1.3 from the above statement, it suffices to observe that for any two points y_1, y_2 of Y:

- the functions $\operatorname{dist}(y_1,\cdot)$ and $\operatorname{dist}(y_2,\cdot)$ differ by at most $\operatorname{dist}(y_1,y_2)$, and
- for any geodesic ray γ starting from y_1 there is a geodesic ray starting from y_2 which stays at a bounded distance from γ.

2.6. Let F be a sequence of μ-preserving transformations of X and $\mathcal{B}$ a family of measurable subsets of X. From Lemma 2.3 it is clear that $\mathcal{B}$ is Borel-Cantelli for F iff for any nonsummable sequence $\mathcal{A}$ of sets from $\mathcal{B}$ one has $S_{\mathcal{A}^F,\infty} = \infty$ for almost all $x \in X$. Therefore we are led to studying asymptotical lower estimates for $S_{\mathcal{H},N}/E_{\mathcal{H},N}$, with $\mathcal{H}$ as in §2.1.

One can easily find many examples of sequences $\mathcal{H}$ for which the above ratio almost surely tends to zero as $N \to \infty$. It is also well known (see [Sp, p. 317] for a historical overview) that the estimates we are after follow from certain conditions on second moments of the functions h_t. We will employ a lemma which was abstracted by V. Sprindžuk from the works of W. Schmidt (see also [P] for a related result).

Lemma ([Spr, Chapter I, Lemma 10]). *For a sequence $\mathcal{H} = \{h_t \mid t \in \mathbb{N}\}$ of functions on X, assume that*

$$\mu(h_t) \leq 1 \quad \textit{for all } t \in \mathbb{N} \tag{2.7}$$

and

$$\text{(SP)}\quad \exists C > 0 \textit{ such that } \int_X \Big(\sum_{t=M}^{N} h_t(x) - \sum_{t=M}^{N} \mu(h_t)\Big)^2 d\mu \leq C\cdot \sum_{t=M}^{N} \mu(h_t) \quad \forall N > M \geq 1\,.$$

Then for any positive ε one has, as $N \to \infty$,

$$S_{\mathcal{H},N}(x) = E_{\mathcal{H},N} + O(E_{\mathcal{H},N}^{1/2} \log^{3/2+\varepsilon} E_{\mathcal{H},N}) \tag{2.8}$$

for μ-a.e. $x \in X$; in particular, $\dfrac{S_{\mathcal{H},N}(x)}{E_{\mathcal{H},N}} \underset{\text{a.e.}}{\longrightarrow} 1$ as $N \to \infty$ whenever $\mathcal{H}$ is nonsummable.

2.7. Remark. Note that the left hand side of (SP) is equal to

$$\int_X \Big(\sum_{t=M}^{N} h_t\Big)^2 d\mu - \Big(\sum_{t=M}^{N} \mu(h_t)\Big)^2 = \sum_{s,t=M}^{N} \big(\mu(h_s h_t) - \mu(h_s)\mu(h_t)\big)\,. \tag{2.9}$$

This shows that (SP) will hold provided the correlation coefficients $|\mu(h_s h_t) - \mu(h_s)\mu(h_t)|$ become small for large values of $|s-t|$. Our plan is to apply Lemma 2.6 to the twisted sequences $\mathcal{H}^F$, where F is as in Theorem 1.8 and $\mathcal{H}$ consists of smooth functions on G/Γ. The exponential decay of correlations under the F-action, the main result of the next section, will be enough to guarantee (SP).

2.8. We close the section with a partial converse to Corollary 2.4, which we will use later for the derivation of logarithm laws.

Lemma. *Let X, μ, d, k, $\mathfrak{d}_+$, $\mathbf{z} \mapsto f_{\mathbf{z}}$, Δ and t_0 be as in Corollary 2.4, and let $r : [t_0, \infty) \mapsto \mathbb{R}$ be a quasi-increasing function such that the integral* (2.2) *diverges. Assume that there exists a lattice Σ in $\mathbb{R}^d$ such that the family $\mathcal{B}(\Delta)$ of super-level sets of Δ is Borel-Cantelli for $F \overset{\text{def}}{=} \{f_{\mathbf{z}} \mid \mathbf{z} \in \mathfrak{d}_+ \cap \Sigma\}$. Then for μ-almost all $x \in X$ there exist $\mathbf{z} \in \mathfrak{d}_+$ arbitrarily far from 0 such that $\Delta\big(f_{\mathbf{z}}(x)\big) \ge r(\|\mathbf{z}\|)$. Consequently,* $\limsup_{\mathbf{z}\in\mathfrak{d}_+,\, \mathbf{z}\to\infty} \dfrac{\Delta\big(f_{\mathbf{z}}(x)\big)}{\log\|\mathbf{z}\|} \underset{\mu\text{-a.e.}}{\ge} d/k\,.$

Proof. From (2.1) and the divergence of (2.2) it follows that the series (2.4) is divergent. In view of Δ being k-DL and by definition of $\mathcal{B}(\Delta)$ being Borel-Cantelli for F, one gets $\Delta\big(f_{\mathbf{z}}(x)\big) \ge r(\|\mathbf{z}\|)$ almost surely for infinitely many $\mathbf{z} \in \mathfrak{d}_+ \cap \Sigma$, hence the first part of the lemma. The second part is immediate by taking $r(t) = \frac{d}{k}\log t$. $\square$

§3. Isolation properties of representations and correlation decay

3.1. Let G be a locally compact second countable group. Recall that the Fell topology on the set of (equivalence classes of) unitary representations ρ of G in separable Hilbert spaces V is defined so that the sets $\{\rho \mid \|\rho(g)v - v\| < \varepsilon\|v\|\ \forall g \in K\ \forall v \in V\}$, where $\varepsilon > 0$ and K runs through all compact subsets of G, constitute a basis of open neighborhoods of the trivial representation I_G of G. (See the Appendix and [M, Chapter III] for more detail.) If (X,μ) is a probability space and $(g,x) \mapsto gx$ a μ-preserving action of G on X, we will denote by $L^2_0(X,\mu)$ the subspace of $L^2(X,\mu)$ orthogonal to constant functions. Our proof of Theorem 1.12 will use the following result, communicated by A. Furman and Y. Shalom, which will allow us to pass from a space to its finite covering:

Lemma. *Let (X_1,μ_1) and (X_2,μ_2) be probability spaces, G a locally compact second countable group acting ergodically on both, and let $\pi : X_1 \mapsto X_2$ be a surjective measurable G-equivariant map such that for some positive $c < 1$ one has*

$$c\mu_1(A) \le \mu_2\big(\pi(A)\big) \le \frac{1}{c}\mu_1(A) \quad \textit{for any } A \subset X_1\,. \tag{3.1}$$

Denote by $\rho_{i,0}$ the regular representation of G on $L^2_0(X_i,\mu_i)$ ($i = 1,2$). Then $\rho_{1,0}$ is isolated from I_G iff so is $\rho_{2,0}$.

The proof of Furman and Shalom is based on the connection between ρ_0 being close to I_G and existence of nontrivial G-invariant means on $L^\infty(X,\mu)$ [FS, Theorem 1.8]. In the Appendix we give a more transparent proof, based on the notion of asymptotically invariant sequences of subsets of X. The argument goes back to J. Rosenblatt [Ro] and K. Schmidt [S] and runs more or less in parallel to the proof given in [FS].

3.2. Let now G be a connected semisimple center-free Lie group without compact factors, Γ an irreducible lattice in G, μ the normalized Haar measure on the homogeneous space G/Γ. It is known (see [B, Lemma 3]) that the regular representation ρ_0 of G on $L^2_0(G/\Gamma,\mu)$ is isolated from I_G. The latter property is also known to be equivalent to the following *spectral gap condition*: there exist a positive lower bound for the spectrum of the Laplacian Δ on $K\backslash G/\Gamma$, where K is a maximal compact subgroup of G.

If G is a direct product of simple groups $G_1,\dots,G_l$, one can decompose Δ as a sum $\Delta_1+\cdots+\Delta_l$, where Δ_i corresponds to coordinates coming from G_i. Then a lower bound for the spectrum of Δ_i amounts to the isolation of $\rho_0|_{G_i}$ from the trivial representation $I|_{G_i}$ of G_i. In the paper [KM] it was implicitly conjectured that restrictions $\rho_0|_{G_i}$ are isolated from $I|_{G_i}$. Theorem 1.12 proves this conjecture in the non-uniform lattice case. The main ingredient of the proof is an explicit bound for the bottom of spectra of Laplacians given by M.-F. Vigneras in [V]. The reduction to the case where these bounds are applicable is based on Lemma 3.1, the Arithmeticity Theorem and the restriction technique of M. Burger and P. Sarnak. We now present the

Proof of Theorem 1.12. If G is simple, the claim follows from [B, Lemma 4.1]. Therefore we can assume that the $\mathbb{R}$-rank of G is greater than 1. By Margulis' Arithmeticity Theorem (see [Z, Theorem 6.1.2] or [M, Chapter IX]), Γ is an arithmetic lattice in G. That is, there exists a semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and a surjective homomorphism $\varphi:\mathbf{G}(\mathbb{R})^0\mapsto G$ such that:

(i) $\operatorname{Ker}\varphi$ is compact, and
(ii) the subgroups $\varphi\big(\mathbf{G}(\mathbb{Z})\cap\mathbf{G}(\mathbb{R})^0\big)$ and Γ are commensurable.

Further, since Γ is non-uniform and G is center-free, $\mathbf{G}$ can be taken to be connected and adjoint, and $\operatorname{Ker}\varphi$ to be trivial (see [Z, Corollary 6.1.10]). By (ii) above, the spaces G/Γ and $G/\varphi\big(\mathbf{G}(\mathbb{Z})\big)$ have a common finite covering. In view of Lemma 3.1, without loss of generality one can assume that $\Gamma=\mathbf{G}(\mathbb{Z})$ and $G=\mathbf{G}(\mathbb{R})$.

Write $\mathbf{G}$ in the form $R_{k/\mathbb{Q}}\tilde{\mathbf{G}}$, where k is a finite extension of $\mathbb{Q}$, $\tilde{\mathbf{G}}$ is an absolutely k-simple k-group, and $R_{k/\mathbb{Q}}$ stands for Weil's restriction of scalars functor (see [T2, 3.1.2]). Namely, $\mathbf{G}=\prod_{i=1}^l\tilde{\mathbf{G}}^{\sigma_i}$, where $\sigma_1,\dots,\sigma_l$ are distinct imbeddings of k into $\mathbb{C}$. This way, factors G_i of G can be identified with $\tilde{\mathbf{G}}^{\sigma_i}(\mathbb{R})$ if σ_i is real, or with $\tilde{\mathbf{G}}^{\sigma_i}(\mathbb{C})$ if σ_i is complex.

Since Γ is non-uniform, $\tilde{\mathbf{G}}$ is isotropic over k (see [M, Theorem I.3.2.4(b)]), therefore (see [T1, 3.1, Proposition 13] or [M, Proposition I.1.6.3]) there exists a k-morphism $\tilde{\alpha}:\mathbf{SL}_2\mapsto\tilde{\mathbf{G}}$ with finite kernel. Denote the $\tilde{\alpha}$-image of $\mathbf{SL}_2$ by $\tilde{\mathbf{H}}$, and let $\mathbf{H}=R_{k/\mathbb{Q}}\tilde{\mathbf{H}}$ and $H=\mathbf{H}(\mathbb{R})$. Clearly to show that $\rho_0|_{G_i}$ is isolated from $I|_{G_i}$, it will be enough to prove that $\rho_0|_{H_i}$ is isolated from $I|_{H_i}$, where H_i are almost simple factors of H, isomorphic to $\tilde{\mathbf{H}}^{\sigma_i}(\mathbb{C})$ for complex imbeddings σ_i and to $\tilde{\mathbf{H}}^{\sigma_i}(\mathbb{R})$ for real ones.

We now use Theorem 1.1 from the paper [BS], which guarantees that $\rho_0|_H$ lies in the closure of the *automorphic spectrum* of H (the latter stands for irreducible components of representations of H on all the spaces $L^2(H/\Lambda)$ where Λ is a congruence subgroup of $\mathbf{H}(\mathbb{Z})$). Denote by $\mathbf{L}$ the algebraic group $R_{k/\mathbb{Q}}\mathbf{SL}_2$ and by α the isogeny $\mathbf{L}\mapsto\mathbf{H}$ induced by $\tilde{\alpha}$. Note that homogeneous spaces H/Λ can be identified with $\mathbf{L}(\mathbb{R})/\alpha^{-1}(\Lambda)$, and preimages of congruence subgroups of $\mathbf{H}(\mathbb{Z})$ are congruence subgroups of $\mathbf{L}(\mathbb{Z})$. Therefore it suffices to check that nontrivial irreducible components of regular representations of almost $\mathbb{R}$-simple factors of $\mathbf{L}(\mathbb{R})$ on $L^2\big(\mathbf{L}(\mathbb{R})/\Lambda\big)$ are uniformly isolated from the trivial representation for all $i=1,\dots,r$ and all principal congruence subgroups Λ of $\mathbf{L}(\mathbb{Z})$. The latter statement is a reformulation of one of the corollaries in Section VI of the paper [V], with the uniform bound for the first nonzero eigenvalue of the corresponding Laplace operators being equal to 3/16 for real and 3/4 for complex imbeddings σ_i. $\square$

3.3. Remark. One can also prove Theorem 1.12 without using Lemma 3.1 by extending the result of Vigneras to arbitrary subgroups of $\mathbf{H}(\mathbb{Z})$ rather than congruence subgroups. For this one can use the centrality of the congruence kernel for higher rank groups, see [R2], and the results of Y. Flicker [F] on lifting of automorphic representations to metaplectic coverings of GL_2. This way it should be possible to get an explicit uniform (in all G and Γ) bound for the neighborhood of the trivial representation which is disjoint from all the restrictions $\rho_0|_{G_i}$.

3.4. We now turn to the paper [KM], where the well-known (from the work of Harish Chandra, Howe, Cowling and Katok-Spatzier) connection between isolation properties of ρ_0 and exponential decay of its matrix coefficients has been made explicit. Let G be a connected semisimple Lie group with finite center, K its maximal compact subgroup. Take an orthonormal basis $\{X_i\}$ of the Lie algebra of K, and denote by Υ the differential operator $1 - \sum_{i=1}^{\dim(K)} X_i^2$ (see [W, §4.4.2]).

Theorem (see [KM, Corollary 2.4.4] and a correction on p. 172). *Let Π be a family of unitary representations of G such that the restriction of Π to any simple factor of G is isolated from the trivial representation. Then there exist a universal constant $B > 0$, a positive integer l (dependent only on G) and $\beta > 0$ (dependent on Π and on the choice of the bi-K-invariant norm $\|g\| = \operatorname{dist}(g,e)$ on G) such that for any $\rho \in \Pi$, any C^∞-vectors v, w in a representation space of ρ and any $g \in G$ one has*

$$\big|\big(\rho(g)v, w\big)\big| \le Be^{-\beta\|g\|}\|\Upsilon^l(v)\|\|\Upsilon^l(w)\|\,. \tag{3.2}$$

Combining Theorem 3.4 and Theorem 1.12, we obtain the following

3.5. Corollary. *Let G be a connected semisimple center-free Lie group without compact factors, Γ an irreducible non-uniform lattice in G, $X = G/\Gamma$, μ the normalized Haar measure on X. Then there exist $B, \beta > 0$ and $l \in \mathbb{N}$ such that for any two functions $\varphi, \psi \in C^\infty(X) \cap L^2(X)$ and any $g \in G$ one has*

$$\big|(g\varphi, \psi) - \mu(\varphi)\mu(\psi)\big| \le Be^{-\beta\|g\|}\|\Upsilon^l(\varphi)\|\|\Upsilon^l(\psi)\|\,. \tag{3.2}$$

Proof. The family $\Pi = \{\rho_0\}$ satisfies the assumption of Theorem 3.4 in view of Theorem 1.12. Therefore one can apply (3.2) to the functions $\varphi - \mu(\varphi)$ and $\psi - \mu(\psi)$. $\square$

§4. A quantitative version of Theorem 1.8

4.1. Let G, Γ and μ be as in Theorem 1.12, and denote the (noncompact) homogeneous space G/Γ by X. Our first goal is to apply Lemma 2.6 to certain sequences of functions on X. For $l \in \mathbb{N}$ and $C > 0$, say that $h \in C^\infty(X) \cap L^2(X)$ is *(C, l)-regular* if

$$\|\Upsilon^l(h)\| \le C \cdot \mu(h)\,.$$

Proposition. *Assume that $F = \{f_t \mid t \in \mathbb{N}\}$ is an ED sequence of elements of G. Take $l \in \mathbb{N}$ as in Corollary 3.5 and an arbitrary $C > 0$, and let $\mathcal{H} = \{h_t\}$ be a sequence of (C,l)-regular functions on X such that* (2.7) *holds. Then the twisted sequence $\mathcal{H}^F$ satisfies* (SP)*; in particular,* (2.8) *holds and*

$$\lim_{N\to\infty} \frac{S_{\mathcal{H}^F,N}(x)}{E_{\mathcal{H},N}} = 1 \quad \textit{for } \mu\textit{-a.e. } x \in X$$

whenever $\mathcal{H}$ is nonsummable.

Proof. In view of (2.9), one has to estimate the sum

$$\sum_{s,t=M}^{N} \big((f_s^{-1}h_s, f_t^{-1}h_t) - \mu(h_s)\mu(h_t)\big) \tag{4.1}$$

from above. Observe that, since μ is F-invariant, $(f_s^{-1}h_s, f_t^{-1}h_t) - \mu(h_s)\mu(h_t)$ is equal to

$$\begin{aligned}(h_s, f_s f_t^{-1}h_t) - \mu(h_s)\mu(h_t) &\underset{\text{(by Corollary 3.5)}}{\le} Be^{-\beta\|f_s f_t^{-1}\|}\|\Upsilon^l(h_s)\|\|\Upsilon^l(h_t)\| \\ \underset{\text{(by the }(C,l)\text{-regularity of } h_s,\ h_t)}{\le} BC^2 e^{-\beta\|f_s f_t^{-1}\|}\mu(h_s)\mu(h_t) &\underset{\text{(by (2.7))}}{\le} BC^2 e^{-\beta\|f_s f_t^{-1}\|}\mu(h_t)\,.\end{aligned}$$

Therefore the sum (4.1) is not bigger than

$$BC^2 \sum_{s,t=M}^{N} e^{-\beta\|f_s f_t^{-1}\|}\mu(h_t) = BC^2 \sum_{t=M}^{N}\mu(h_t)\sum_{s=M}^{N} e^{-\beta\|f_s f_t^{-1}\|} \le BC^2\cdot\sup_{t\in\mathbb{N}}\sum_{s=1}^{\infty} e^{-\beta\|f_s f_t^{-1}\|}\cdot E_N\,.$$

In view of (ED), the constant in the right hand side is finite, and (SP) follows; the "in particular" part is then immediate from Lemma 2.6. $\square$

4.2. Let now Δ be a DL function on X. Similarly to (1.4), for $z \in \mathbb{R}$ we will denote by $A(z)$ the set $\{x \in X \mid \Delta(x) \ge z\}$ (note that it follows from (DL) that $A(z)$ is never empty). To prove a quantitative strengthening of Theorem 1.8 that we are after, we need to learn how to approximate the sets $A(z)$ by smooth functions.

Lemma. *Let Δ be a DL function on X. Then for any $l \in \mathbb{N}$ there exists $C > 0$ such that for every $z \in \mathbb{R}$ one can find two (C,l)-regular nonnegative functions h' and h'' on X such that*

$$h' \le 1_{A(z)} \le h'' \quad \textit{and} \quad c\cdot\mu\big(A(z)\big) \le \mu(h') \le \mu(h'') \le \frac{1}{c}\mu\big(A(z)\big)\,, \tag{4.2}$$

with c as in (DL).

Proof. For $\varepsilon > 0$, let us denote by $A'(z,\varepsilon)$ the set of all points of $A(z)$ which are not ε-close to $\partial A(z)$, i.e. $A'(z,\varepsilon) \overset{\text{def}}{=} \{x \in A(z) \mid \operatorname{dist}\big(x, \partial A(z)\big) \ge \varepsilon\}$, and by $A''(z,\varepsilon)$ the ε-neighborhood of $A(z)$, i.e. $A''(z,\varepsilon) \overset{\text{def}}{=} \{x \in X \mid \operatorname{dist}\big(x, A(z)\big) \le \varepsilon\}$. (If $A(z) = X$, the above sets will coincide with X.)

Choose δ and c according to (DL). Then, using the uniform continuity of Δ, find $\varepsilon > 0$ such that

$$|\Delta(x) - \Delta(y)| < \delta \text{ whenever } \operatorname{dist}(x,y) < \varepsilon\,. \tag{4.3}$$

From (4.3) it immediately follows that for all z one has $A(z+\delta) \subset A'(z,\varepsilon) \subset A''(z,\varepsilon) \subset A(z-\delta)$, therefore one can apply (DL) to conclude that

$$c\cdot\mu\big(A(z)\big) \le \mu\big(A'(z,\varepsilon)\big) \le \mu\big(A''(z,\varepsilon)\big) \le \frac{1}{c}\mu\big(A(z)\big)\,. \tag{4.4}$$

Now take a nonnegative $\psi \in C^\infty(G)$ of L^1-norm 1 such that $\operatorname{supp}(\psi)$ belongs to the ball of radius $\varepsilon/4$ centered in $e \in G$. Fix $z \in \mathbb{R}$ and consider functions $h' \stackrel{\text{def}}{=} \psi * 1_{A'(z,\varepsilon/2)}$ and $h'' \stackrel{\text{def}}{=} \psi * 1_{A''(z,\varepsilon/2)}$. Then one clearly has

$$1_{A'(z,\varepsilon)} \le h' \le 1_{A(z)} \le h'' \le 1_{A''(z,\varepsilon)}\,,$$

which, together with (4.4), immediately implies (4.2). It remains to observe that $\|\Upsilon^l h'\| = \|\Upsilon^l(\psi * 1_{A'(z,\varepsilon/2)})\| = \|\Upsilon^l(\psi) * 1_{A'(z,\varepsilon/2)}\|$, so by the Young inequality,

$$\|\Upsilon^l h'\| \le \|\Upsilon^l(\psi)\| \cdot \mu\big(A'(z,\varepsilon/2)\big) \le \|\Upsilon^l(\psi)\| \cdot \mu\big(A(z)\big) \underset{(4.2)}{\le} \frac{1}{c}\|\Upsilon^l(\psi)\|\mu(h') \quad \text{for any } l \in \mathbb{N}\,.$$

Similarly $\|\Upsilon^l h''\| \le \|\Upsilon^l(\psi)\| \cdot \mu\big(A''(z,\varepsilon/2)\big) \underset{(4.4)}{\le} \|\Upsilon^l(\psi)\| \cdot \frac{1}{c}\mu\big(A(z)\big) \le \frac{1}{c}\|\Upsilon^l(\psi)\| \cdot \mu(h'')$, hence, with $C = \frac{1}{c}\|\Upsilon^l(\psi)\|$, both h' and h'' are (C,l)-regular, and the lemma is proven. $\square$

4.3. We now state and prove the promised quantitative strengthening of Theorem 1.8.

Theorem. *Let G, Γ, $F = \{f_t\}$ and Δ be as in Theorem 1.8, and let $\{r_t\}$ be a sequence of real numbers such that*

$$\sum_{t=1}^{\infty} \Phi_\Delta(r_t) = \infty\,. \tag{4.5}$$

Then for some positive $c \le 1$ and for almost all $x \in G/\Gamma$ one has

$$c \le \liminf_{N\to\infty} \frac{\#\{1 \le t \le N \mid \Delta(f_t x) \ge r_t\}}{\sum_{t=1}^N \Phi_\Delta\big(r_t\big)} \le \limsup_{N\to\infty} \frac{\#\{1 \le t \le N \mid \Delta(f_t x) \ge r_t\}}{\sum_{t=1}^N \Phi_\Delta\big(r_t\big)} \le \frac{1}{c}\,.$$

It is clear that Theorem 1.8 is a direct consequence of Lemma 2.3 and the first of the above inequalities. Note that D. Sullivan proved that in the setting of Theorem 1.2 one has a positive lower bound for

$$\limsup_{N\to\infty} \frac{\#\{1 \le t \le N \mid \operatorname{dist}\big(y_0, \gamma_t(y,\xi)\big) \ge r_t\}}{\sum_{t=1}^N e^{-kr_t}}$$

for almost all $\xi \in S_y(Y)$ (see [Su, §9, Remark (2)]).

Proof. First let us assume that the center of G is trivial; after that we will reduce the general case to the center-free situation. Observe that from the existence of a DL function Δ on X it follows that X can not be compact: indeed, Δ must be uniformly continuous, but unbounded in view of (DL). Take l as in Corollary 3.5 and C as in Lemma 4.2. For any $t \in \mathbb{N}$, let h_t' and h_t'' stand for the (C,l)-regular functions which one associates with the set $A(r_t) = \{x \in X \mid \Delta(x) \ge r_t\}$ by means of Lemma 4.2, and let us denote

$$\mathcal{A} = \{A(r_t) \mid t \in \mathbb{N}\}\,, \quad \mathcal{H}' = \{h_t' \mid t \in \mathbb{N}\}\,, \quad \mathcal{H}'' = \{h_t'' \mid t \in \mathbb{N}\}\,.$$

By (4.5), the sequence $\mathcal{A}$ is nonsummable; hence, in view of (4.2), the same can be said about $\mathcal{H}'$ and $\mathcal{H}''$. Also it is clear from the construction that $\mu(h_t') \le \mu(h_t'') \le 1$. Therefore, as $N \to \infty$, by Proposition 4.1 the ratios $S_{(\mathcal{H}')^F,N}(x)/E_{\mathcal{H}',N}$ and $S_{(\mathcal{H}'')^F,N}(x)/E_{\mathcal{H}'',N}$ tend

to 1 for μ-a.e. $x \in X$. But from (4.2) it follows that $S_{(\mathcal{H}')^F,N} \le S_{\mathcal{A}^F,N} \le S_{(\mathcal{H}'')^F,N}$ and $\frac{1}{c}E_{\mathcal{H}',N} \ge E_{\mathcal{A},N} \ge c \cdot E_{\mathcal{H}'',N}$ for any $N \in \mathbb{N}$. Therefore μ-almost everywhere one has

$$c = \lim_{N\to\infty} \frac{S_{(\mathcal{H}')^F,N}(x)}{\frac{1}{c}E_{\mathcal{H}',N}} \le \liminf_{N\to\infty} \frac{S_{\mathcal{A}^F,N}(x)}{E_{\mathcal{A},N}} \le \limsup_{N\to\infty} \frac{S_{\mathcal{A}^F,N}(x)}{E_{\mathcal{A},N}} \le \lim_{N\to\infty} \frac{S_{(\mathcal{H}'')^F,N}(x)}{c \cdot E_{\mathcal{H}'',N}} = \frac{1}{c}\,,$$

and the statement of the theorem follows.

Now let us look at what happens if G has nontrivial center Z. Let us denote the quotient group G/Z by G', the homomorphism $G \mapsto G'$ by p, and the induced map $X \mapsto X' \stackrel{\text{def}}{=} G'/p(\Gamma)$ by $\bar{p}$. Since ΓZ is discrete [R1, Corollary 5.17], $p(\Gamma)$ is also discrete, hence $Z/(\Gamma \cap Z)$ is finite. This means that $(X, \bar{p})$ is a finite covering of X'; moreover, one can choose representatives $g_1, \dots, g_l$ ($g_1 = e$) from cosets of $Z/(\Gamma \cap Z)$ which will act isometrically on X. In particular, the distance between $x \in X$ and $g_i x$, $1 \le i \le l$, is uniformly bounded by some constant C. Now, given a DL function Δ on X, define Δ' on X' by $\Delta'\big(\bar{p}(x)\big) = \frac{1}{l}\sum_{y \in \bar{p}^{-1}\circ\bar{p}(x)} \Delta(y) = \frac{1}{l}\sum_{i=1}^{l} \Delta(g_i x)$. Then from the uniform continuity of Δ it follows that Δ' is also uniformly continuous, and for some constant C' one has

$$|\Delta'\big(\bar{p}(x)\big) - \Delta(x)| \le C' \quad \forall\, x \in X\,. \tag{4.6}$$

Therefore for any $z > 0$, $\Phi_{\Delta'}(z) = \mu\big(\{x \in X \mid \Delta'\big(\bar{p}(x)\big) \ge z\}\big)$ is bounded between $\Phi_\Delta(z + C')$ and $\Phi_\Delta(z - C')$. This implies that Δ' satisfies (DL) as well; moreover,

$$\frac{\Phi_{\Delta'}(z)}{\Phi_\Delta(z)} \text{ is uniformly bounded between two positive constants.} \tag{4.7}$$

Finally, assume that (4.5) holds and $F \subset G$ is ED. It follows that $\{p(F)\}$ is also ED, and from (4.7) one deduces that $\sum_{t=1}^{\infty} \Phi_{\Delta'}(r_t) = \infty$ as well. Therefore one can use the center-free case of Theorem 4.3 and Δ' being a DL function to conclude that for some $0 < c \le 1$ and for μ-almost all $x \in X$ one has

$$c \le \liminf_{N\to\infty} \frac{\#\{1 \le t \le N \mid \Delta'\big(\bar{p}(f_t x)\big) \ge r_t + C'\}}{\sum_{t=1}^{N} \Phi_{\Delta'}(r_t)}$$

and

$$\limsup_{N\to\infty} \frac{\#\{1 \le t \le N \mid \Delta'\big(\bar{p}(f_t x)\big) \ge r_t - C'\}}{\sum_{t=1}^{N} \Phi_{\Delta'}(r_t)} \le \frac{1}{c}\,.$$

Clearly (4.6) implies that

$$\Delta'\big(\bar{p}(f_t x)\big) \ge r_t + C' \quad \Rightarrow \quad \Delta(f_t x) \ge r_t \quad \Rightarrow \quad \Delta'\big(\bar{p}(f_t x)\big) \ge r_t - C'\,.$$

Therefore to finish the proof it remains to replace the values of $\Phi_{\Delta'}$ by those of Φ_Δ, sacrificing no more than a multiplicative constant in view of (4.7). $\square$

4.4. *Proof of Theorems 1.7 and 1.9.* Recall that in part (a) of Theorem 1.9 we are given a sequence $F = \{f_t\} = \{\exp(\mathbf{z}_t)\}$ such that (1.6) holds. It is easy to check that F satisfies (ED): for any $\beta > 0$ one can write

$$\sup_{t\in\mathbb{N}} \sum_{s=1}^{\infty} e^{-\beta\|f_s f_t^{-1}\|} = \sup_{t\in\mathbb{N}} \sum_{s=1}^{\infty} e^{-\beta\|\mathbf{z}_s - \mathbf{z}_t\|} \le \sup_{t\in\mathbb{N}} \sum_{n=0}^{\infty} e^{-\beta n}\#\{s \mid n \le \|\mathbf{z}_s - \mathbf{z}_t\| \le n+1\}$$

$$\underset{(1.6)}{\le} \text{const} \cdot \sum_{n=0}^{\infty} n^{\dim(\mathfrak{a})} e^{-\beta n} < \infty\,.$$

Therefore Theorem 1.8 applies and one concludes that $\mathcal{B}(\Delta)$ is Borel-Cantelli for F. Part (b) is then immediate from Corollary 2.4 and Lemma 2.8. It remains to notice that Theorem 1.7 is a special case of Theorem 1.9, with $\mathbf{z}_t = t\mathbf{z}$, $d = 1$, $\mathfrak{d} = \mathbb{R}\mathbf{z}$ and $\mathfrak{d}_+ = \{t\mathbf{z} \mid t \ge 0\}$. $\square$

§5. Distance functions are DL

5.1. The goal of the section is to prove the following

Proposition. *Let G be a connected semisimple Lie group, Γ a non-uniform irreducible*[5] *lattice in G, K a maximal compact subgroup of G, μ the normalized Haar measure on G/Γ, x_0 a point in G/Γ, $\operatorname{dist}(\cdot,\cdot)$ a Riemannian metric on G/Γ chosen by fixing a right invariant Riemannian metric on G bi-invariant with respect to K. Then there exists $k > 0$ such that the function $\operatorname{dist}(x_0,\cdot)$ is k-DL.*

5.2. Remark. Let (X_1, x_1) and (X_2, x_2) be pointed metric spaces with probability measures μ_1 and μ_2, and let $\pi : X_1 \mapsto X_2$ be a measurable surjective map which almost preserves distances from base points (i.e. with $\sup_{x\in X_1} \big|\operatorname{dist}(x_1, x) - \operatorname{dist}\big(x_2, \pi(x)\big)\big| < \infty$) and satisfies the following property: for some positive $c < 1$ one has

$$c\mu_2(A) \le \mu_1\big(\pi^{-1}(A)\big) \le \frac{1}{c}\mu_2(A) \quad \text{for any } A \subset X_2\,.$$

Then the function $\operatorname{dist}(x_1,\cdot)$ on X_1 is k-DL iff so is $\operatorname{dist}(x_2,\cdot)$ on X_2. This observation will be used many times in the proof below, sometimes without explicit mention. Examples include:

- $X_1 = X_2$, $\mu_1 = \mu_2$ (shift of base point);
- $X_1 = X_2 \times Q$ (the direct product of metric and probability spaces), π the projection on X_2, $\operatorname{diam}(Q) < \infty$;
- $X_1 \overset{\pi}{\mapsto} X_2$ a finite covering, $\mu_2 = \pi(\mu_1)$.

5.3. *Proof of Proposition 5.1.* First suppose that the $\mathbb{R}$-rank of G is greater than 1. Then, using the Arithmeticity Theorem, as in the proof of Theorem 1.12 (see §3.2) we can assume that $G = \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a semisimple algebraic $\mathbb{Q}$-group and Γ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$.

We now need to use the reduction theory for arithmetic groups. Let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$. Denote the identity component of $\mathbf{T}(\mathbb{R})$ by A, and its Lie algebra by $\mathfrak{a}$. Let Φ be a system of $\mathbb{Q}$-roots associated with $\mathfrak{a}$. Choose an ordering of Φ, let Φ^+ (resp. Φ^s) be the set of positive (resp. simple) roots, and let $\mathfrak{a}_+$ stand for the closed $\mathbb{Q}$-Weyl chamber in $\mathfrak{a}$ defined by $\mathfrak{a}_+ \overset{\text{def}}{=} \{\mathbf{z} \in \mathfrak{a} \mid \alpha(\mathbf{z}) \ge 0 \;\forall\, \alpha \in \Phi^s\}$. We set $A_+ \overset{\text{def}}{=} \exp(\mathfrak{a}_+)$.

Let $G = KAMU$ be a (generalized) Iwasawa decomposition for G, where K is maximal compact in G, U is unipotent and M is reductive (here A centralizes M and normalizes U). Then one defines a *generalized Siegel set* $\mathcal{S}_{Q,\tau}$ as follows: $\mathcal{S}_{Q,\tau} \overset{\text{def}}{=} K\exp(\mathfrak{a}_\tau)Q$, where Q is relatively compact in MU, $\tau \in \mathbb{R}$ and $\mathfrak{a}_\tau \overset{\text{def}}{=} \{\mathbf{z} \in \mathfrak{a} \mid \alpha(\mathbf{z}) \ge \tau \;\forall\, \alpha \in \Phi^s\}$. It is known that a finite union of translates of such a set (for suitable Q and τ) forms a *weak fundamental set* for the G-action on G/Γ. More precisely, the following is true:

5.4. Theorem ([Bo, §13] or [L, Proposition 2.2]). *Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-group and Γ an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then there exist a generalized Siegel set $\mathcal{S} = \mathcal{S}_{Q,\tau} \subset G = \mathbf{G}(\mathbb{R})$ and $\{q_1,\dots,q_m\} \subset \mathbf{G}(\mathbb{Q})$ such that the union $\Omega \overset{\text{def}}{=} \cup_{i=1}^m \mathcal{S}q_i$ satisfies the following two properties:*

(i) *$G = \Omega\Gamma$;*
(ii) *for any $q \in \mathbf{G}(\mathbb{Q})$, the set $\{\gamma \in \Gamma \mid \Omega q \cap \Omega\gamma\}$ is finite.*

[5] Again, the proposition is also true for reducible lattices, see §10.2.

In other words, the restriction to Ω of the natural projection π of G onto G/Γ is surjective and at most finite-to-one.

We now want to study metric properties of the restriction $\pi|_\Omega$. Since the distance on G/Γ is defined by $\operatorname{dist}_{G/\Gamma}\big(\pi(x),\pi(y)\big) = \inf_{\gamma\in\Gamma}\operatorname{dist}_G(x,y\gamma)$, one clearly has $\operatorname{dist}_{G/\Gamma}\big(\pi(x),\pi(y)\big) \le \operatorname{dist}_G(x,y)$ for any $x,y\in G$. The converse estimate, with x,y taken from a Siegel set, has been known as Siegel's Conjecture. Its proof is due to J. Ding for $G = SL_n(\mathbb{R})$ and to E. Leuzinger and L. Ji (independently) for the general case. Specifically, the following statement has been proved:

5.5. Theorem ([L, Theorem 5.7] or [J, Theorem 7.6]). *For* $\mathbf{G}$*,* Γ*,* $\mathcal{S}$ *and* $\{q_1,\dots,q_m\}$ *as in Theorem 5.4, there exists a positive constant* D *such that*

$$\operatorname{dist}_G\big(xq_i, yq_j\gamma\big) \ge \operatorname{dist}_G(x,y) - D$$

for all $i,j = 1,\dots,m$*,* $\gamma\in\Gamma$ *and* $x,y\in\mathcal{S}$*.*

In view of the last two theorems and Remark 5.2, it is enough to prove that the function $\operatorname{dist}_G(x_0,\cdot)$ on Ω is k-DL for some $k>0$ and $x_0\in\Omega$ (with respect to suitably scaled Haar measure). Further, since the metric on G is right invariant, it suffices to consider just one copy $\mathcal{S} = K\exp(\mathfrak{a}_\tau)Q$ of the Siegel set instead of the union Ω of several translates thereof.

Our next goal is to reduce the problem to the restriction of the distance function to $\exp(\mathfrak{a}_\tau)$. Since the metric on G is right invariant and bi-K-invariant, the projection $G = KAMU \mapsto A$ is almost distance preserving (in the sense of Remark 5.2). Furthermore, cf. [Bou1, Ch. VII, §9, Proposition 13], the Haar measure on G is being sent to the measure $\delta(a)\,da$, where da is a Haar measure on A and δ is the restriction of the modular function of the group AMU to A. Put differently, $\delta(a)$ is the modulus of the automorphism $x\mapsto axa^{-1}$ of MU (equivalently, of U, since M is centralized by A). Therefore, if $a = \exp(\mathbf{z})$, $\mathbf{z}\in\mathfrak{a}$, the value of δ at a is equal to $e^{\operatorname{tr}(-\operatorname{ad}\mathbf{z})} = e^{-\rho(\mathbf{z})}$, where $\rho \stackrel{\text{def}}{=} \sum_{\alpha\in\Phi^+}\alpha$ is the sum of the positive roots. Since the metric on A is carried from $\mathfrak{a}$ by the exponential map, it suffices to find k such that the function $\mathbf{z}\mapsto\|\mathbf{z}\|$ on $\mathfrak{a}_\tau$ (equivalently, on $\mathfrak{a}_+$, since $\mathfrak{a}_\tau$ is an isometric translate of the latter) is k-DL with respect to the measure $\text{const}\cdot e^{-\rho(\mathbf{z})}\,d\mathbf{z}$.

Let $\{\alpha_1,\dots,\alpha_n\}$ be the simple roots, and $\{\omega_1,\dots,\omega_n\}$ the dual system of fundamental weights (that is, with $\alpha_i(\omega_j) = \delta_{ij}$, $i,j = 1,\dots,n$). One can write

$$\rho = \sum_{i=1}^n k_i\alpha_i\,, \tag{5.1}$$

where k_i are positive integers. The following lemma is what one needs to complete the proof:

5.6. Lemma. *Let* $k = \min_{i=1,\dots,n}\frac{k_i}{\|\omega_i\|}$*. Then there exist* $C_1, C_2 > 0$ *such that for any* $z>0$*, the ratio of* $\int_{\{\mathbf{z}\in\mathfrak{a}_+,\,\|\mathbf{z}\|\ge z\}} e^{-\rho(\mathbf{z})}\,d\mathbf{z}$ *and* e^{-kz} *is bounded between* C_1 *and* C_2*.*

Proof. Without loss of generality assume that $\frac{k_i}{\|\omega_i\|}$ is equal to k for $1\le i\le j$ and is strictly bigger than k for $i>j$. For $r>0$, denote by Σ_r the intersection of $\mathfrak{a}_+$ and the sphere of radius r centered at the origin. This is a spherical simplex with extremal points (vertices) given by $\mathbf{z}_i \stackrel{\text{def}}{=} \frac{r}{\|\omega_i\|}\omega_i$. From the strict convexity of the ball it follows that $\rho|_{\Sigma_r}$ attains its minimal value kr at the points $\mathbf{z}_i$, $1\le i\le j$. Furthermore, one can choose $\varepsilon,\varepsilon',c>0$ such that uniformly in $r>0$ the set

$$\Sigma_{r,\varepsilon} \stackrel{\text{def}}{=} \{\mathbf{z}\in\Sigma_r,\ \rho(\mathbf{z}) \le (k+\varepsilon)r\}$$

belongs to the union of $\varepsilon' r$-neighborhoods of the points $\mathbf{z}_i$, $1 \le i \le j$, and on each of these neighborhoods one has $\rho(\mathbf{z}) - kr \ge c\|\mathbf{z} - \mathbf{z}_i\|$.

Denote by σ the induced Lebesgue measure on Σ_r. Clearly to establish the desired upper estimate for

$$\int_{\{\mathbf{z}\in\mathfrak{a}_+,\ \|\mathbf{z}\|\ge z\}} e^{-\rho(\mathbf{z})}\, d\mathbf{z} = \int_z^\infty \int_{\Sigma_r} e^{-\rho(\mathbf{z})}\, d\sigma(\mathbf{z})\, dr$$

it suffices to prove that $\int_{\Sigma_r} e^{-\rho(\mathbf{z})}\, d\sigma(\mathbf{z})$ is not greater than[6] const$\cdot e^{-kr}$. The latter inequality follows since

$$\begin{aligned}
\int_{\Sigma_r} e^{-\rho(\mathbf{z})}\, d\sigma(\mathbf{z}) &\le \int_{\Sigma_r \smallsetminus \Sigma_{r,\varepsilon}} e^{-\rho(\mathbf{z})}\, d\sigma(\mathbf{z}) + \int_{\Sigma_{r,\varepsilon}} e^{-\rho(\mathbf{z})}\, d\sigma(\mathbf{z}) \\
&\le \int_{\Sigma_r} e^{-(k+\varepsilon)r}\, d\sigma(\mathbf{z}) + \sum_{i=1}^{j} \int_{\{\mathbf{z}\in\Sigma_r,\ \|\mathbf{z}-\mathbf{z}_i\|\le\varepsilon' r\}} e^{-(kr + c\|\mathbf{z}-\mathbf{z}_i\|)}\, d\sigma(\mathbf{z}) \\
&\le \mathrm{const}\cdot r^{n-1} e^{-(k+\varepsilon)r} + \mathrm{const}\cdot e^{-kr} \int_{\mathbb{R}^{n-1}} e^{-c\|\mathbf{x}\|}\, d\mathbf{x} \le \mathrm{const}\cdot e^{-kr}\,.
\end{aligned}$$

As for the lower estimate, the set $\{\mathbf{z} \in \mathfrak{a}_+,\ \|\mathbf{z}\| \ge z\}$ clearly contains the translate $\mathbf{z}_1 + \mathfrak{a}_+$ of $\mathfrak{a}_+$, where, as before, $\mathbf{z}_1 = \frac{z}{\|\omega_1\|}\omega_1$ and $\rho(\mathbf{z}_1) = kz$; therefore

$$\int_{\{\mathbf{z}\in\mathfrak{a}_+,\ \|\mathbf{z}\|\ge z\}} e^{-\rho(\mathbf{z})}\, d\mathbf{z} \ge \int_{\mathbf{z}_1+\mathfrak{a}_+} e^{-\rho(\mathbf{z})}\, d\mathbf{z} = \int_{\mathfrak{a}_+} e^{-\rho(\mathbf{z}+\mathbf{z}_1)}\, d\mathbf{z} = e^{-kz} \int_{\mathfrak{a}_+} e^{-\rho(\mathbf{z})}\, d\mathbf{z}\,,$$

which finishes the proof. $\square$

To complete the proof of Proposition 5.1 it remains to observe that in the case when the $\mathbb{R}$-rank of G is equal to 1, the proof can be written along the same lines, by means of the description [GR] of fundamental domains for lattices in rank-one groups. $\square$

5.7. Note that the above proof, via Lemma 5.6, provides a constructive way to express the exponent k for any homogeneous space G/Γ via parameters of the corresponding system Φ of $\mathbb{Q}$-roots. For example, if $G = SL_n(\mathbb{R})$ and the metric on G is given by the Killing form, one can compute (using e.g. [Bou2, Planche I]) the norms of fundamental weights $\omega_1, \dots, \omega_{n-1}$:

$$\|\omega_i\|^2 = \frac{i(n-i)}{n^2}\big(n(n+1) - 2i(n-i)\big)\,,$$

and the coefficients k_i in (5.1): $k_i = \dfrac{i(n-i)}{2}$. It follows that the ratio

$$\frac{\|\omega_i\|^2}{k_i^2} = \frac{4}{n^2}\left(\frac{n(n+1)}{i(n-i)} - 2\right)$$

attains its maximum when $i = 1$ or $n-1$; therefore $k = \dfrac{k_1}{\|\omega_1\|} = \dfrac{n}{2}\sqrt{\dfrac{n-1}{n^2-n+2}}$. Similar computation can be done for root systems of other types.

[6] The values of constants in the proof below are independent on r.

§6. Geodesics and flats in locally symmetric spaces

6.1. We are now going to use the result of the previous section and derive Theorems 1.4 and 1.10 from Theorems 1.7 and 1.9 respectively. Throughout the end of the section, $Y \cong K\backslash G/\Gamma$ is a noncompact irreducible locally symmetric space of noncompact type and finite volume. Here G is the connected component of the identity in the isometry group of the universal cover $\tilde{Y}$ of Y, Γ an irreducible lattice in G and K a maximal compact subgroup of G, i.e. the stabilizer of a point $\tilde{y}_0 \in \tilde{Y}$.

Denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of G (resp. K). The geodesic symmetry at $\tilde{y}_0$ induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and one can identify the tangent space to a point $\tilde{y}_0 \in Y$ with $\mathfrak{p}$. Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Let $\mathfrak{a}_+$ be a positive Weyl chamber relative to a fixed ordering of the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Then it is known that the set $\mathfrak{a}_1$ of unit vectors in $\mathfrak{a}_+$ is a fundamental set for the G-action on the unit tangent bundle $S(\tilde{Y})$ of $\tilde{Y}$; that is, every orbit of G intersects the set $\{(\tilde{y}_0, \mathbf{z}) \mid \mathbf{z} \in \mathfrak{a}_1\}$ exactly once. Furthermore, for $\mathbf{z} \in \mathfrak{a}_1$, the stabilizer of $(\tilde{y}_0, \mathbf{z})$ in G is the centralizer $K_{\mathbf{z}}$ of $\mathbf{z}$ in K, so the G-orbit of $(\tilde{y}_0, \mathbf{z})$ in $S(\tilde{Y})$ (resp. the G-orbit $\mathcal{E}_{\mathbf{z}} \stackrel{\text{def}}{=} G(y_0, \mathbf{z})$ of $(y_0, \mathbf{z})$ in $S(Y)$) can be identified with $K_{\mathbf{z}}\backslash G$ (resp. with $K_{\mathbf{z}}\backslash G/\Gamma$). The sets $\mathcal{E}_{\mathbf{z}}$, $\mathbf{z} \in \mathfrak{a}_1$, are smooth submanifolds of $S(Y)$ of finite Riemannian volume, which form a singular measurable foliation of $S(Y)$. It will be convenient to introduce the notation $\mathcal{E}_{\mathbf{z},y}$ for the set of all $\xi \in S_y(Y)$ for which $(y, \xi) \in \mathcal{E}_{\mathbf{z}}$ (here y is an arbitrary point of Y). Note that if the $\mathbb{R}$-rank of G is equal to 1, the set $\mathfrak{a}_1$ consists of a single element $\mathbf{z}$, so one has $\mathcal{E}_{\mathbf{z}} = S(Y)$ and $\mathcal{E}_{\mathbf{z},y} = S_y(Y)$ for any $y \in Y$.

It has been shown by F. Mautner [Ma] that the geodesic flow γ_t on $S(Y)$ restricted to $\mathcal{E}_{\mathbf{z}}$, $\mathbf{z} \in \mathfrak{a}_1$, can be described via the action of the one-parameter subgroup $\{\exp(t\mathbf{z})\}$ of G as follows:

$$\gamma_t(K_{\mathbf{z}} g\Gamma) = K_{\mathbf{z}} \exp(t\mathbf{z}) g\Gamma \,. \tag{6.1}$$

This clearly provides a link between Theorems 1.4 and 1.7. In particular, one can prove the following strengthening of Theorem 1.4:

6.2. Theorem. *There exists $k = k(Y) > 0$ such that for any $\mathbf{z} \in \mathfrak{a}_1$ the following holds: if $y_0 \in Y$ and $\{r_t \mid t \in \mathbb{N}\}$ is a sequence of real numbers, then for any $y \in Y$ and almost every (resp. almost no) $\xi \in \mathcal{E}_{\mathbf{z},y}$ there are infinitely many $t \in \mathbb{N}$ such that* (1.2) *is satisfied, provided the series $\sum_{t=1}^{\infty} e^{-kr_t}$ diverges (resp. converges).*

Proof. Let p denote the natural projection from $X = G/\Gamma$ onto $\mathcal{E}_{\mathbf{z}}$, take $x_0 \in p^{-1}(y_0)$ and denote by Δ the function $\operatorname{dist}_X(x_0, \cdot)$ on X. Using Proposition 5.1, find k such that Δ is k-DL. If $\sum_{t=1}^{\infty} e^{-kr_t} = \infty$, then, by Theorem 1.7, for any $C > 0$ and almost all $x \in X$ there are infinitely many $t \in \mathbb{N}$ such that $\Delta\big(\exp(t\mathbf{z})x\big) \geq r_t + C$. But clearly $\Delta(x)$ and $\operatorname{dist}_Y(y_0, y)$ differ by no more than additive constant whenever $p(x) = (y, \xi)$. Therefore it follows from (6.1) that the set

$$\{(y, \xi) \in \mathcal{E}_{\mathbf{z}} \mid \ (1.2) \text{ holds for infinitely many } t \in \mathbb{N}\} \tag{6.2}$$

has full measure in $\mathcal{E}_{\mathbf{z}}$. To finish the proof of the divergence case, it remains to notice that for any $y, y' \in Y$ and $\xi \in \mathcal{E}_{\mathbf{z},y}$ there exists $\xi' \in \mathcal{E}_{\mathbf{z},y'}$ such that $\operatorname{dist}\big(\gamma_t(y, \xi), \gamma_t(y', \xi')\big)$ is uniformly bounded from above for all positive t. Therefore for any $y \in Y$ the intersection of the set (6.2) with $\mathcal{E}_{\mathbf{z},y}$ has full measure in the latter set. The proof of the easier convergence case follows the same pattern (and certainly it suffices to use Lemma 2.3 instead of the full strength of Theorem 1.7). $\square$

6.3. *Proof of Theorem 1.4.* The main statement is a direct consequence of the above theorem and the decomposition of the volume measures on the spheres $S_y(Y)$ in terms of the measures on the leaves $\mathcal{E}_{\mathbf{z},y}$ for all $z \in \mathfrak{a}_1$. As for the logarithm law (1.3), its validity for the set of pairs (y,ξ) of full measure in $S(Y)$ immediately follows from Corollary 2.4 and Lemma 2.8, and then, as in the above proof, one shows that the intersection of this set with $S_y(Y)$ has full measure in $S_y(Y)$ for any $y \in Y$. $\square$

6.4. *Proof of Theorem 1.10* can be written along the same lines, with minor modifications. One considers the G-action on the bundle $S^d(\tilde{Y})$ and finds a representative $(\mathbf{z}_1,\dots,\mathbf{z}_d)$, with $\mathbf{z}_i \in \mathfrak{a}$, in any G-orbit (recall that $\mathfrak{p} \supset \mathfrak{a}$ has been identified with the tangent space to $\tilde{Y}$ at $\tilde{y}_0$). Then G-orbits in $S^d(Y)$ are identified with quotients of $X = G/\Gamma$ by centralizers $K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}$ in K of appropriate ordered d-tuples $(\mathbf{z}_1,\dots,\mathbf{z}_d)$. Similarly to (6.1), one describes $\gamma_{\mathbf{t}}(K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}g\Gamma)$, where $\mathbf{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$, via the action of $\exp(\sum_i t_i\mathbf{z}_i)$ on $K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}\backslash X$. An application of Theorem 1.9 to the $\mathfrak{a}$-action on X then provides the desired dichotomy, hence a logarithm law, for almost all $(y,\vec{\xi})$ in any G-orbit. To derive a similar result for almost every $\vec{\xi} \in S^d_y(Y)$ and any $y \in Y$, one needs to decompose $\mathfrak{a}$ as a union of Weyl chambers $\mathfrak{a}_j$ and, accordingly, break the flat $\mathcal{F} = \{\gamma_{\mathbf{t}}(K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}g\Gamma) \mid \mathbf{t} \in \mathfrak{d}_+\} = \{K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}\exp(\sum_i t_i\mathbf{z}_i)g\Gamma \mid \mathbf{t} \in \mathfrak{d}_+\}$ into pieces $\mathcal{F}_j = \{K_{(\mathbf{z}_1,\dots,\mathbf{z}_d)}\exp(\sum_i t_i\mathbf{z}_i)g\Gamma \mid \mathbf{t} \in \mathfrak{d}_+,\ \sum_i t_i\mathbf{z}_i \in \mathfrak{a}_j\}$. After that it remains to notice that given each of the pieces $\mathcal{F}_j$ and a point $y \in Y$, one can use Iwasawa decomposition for G to find a similar piece $\mathcal{F}'_j$ starting from y which lies at a bounded distance from $\mathcal{F}_j$. $\square$

§7. A very important DL function on the space of lattices

7.1. We now consider another class of examples of DL functions on homogeneous spaces. Throughout the section we fix an integer $k > 1$, let $G = SL_k(\mathbb{R})$, $\Gamma = SL_k(\mathbb{Z})$ and μ the normalized Haar measure on the space $X_k \stackrel{\text{def}}{=} G/\Gamma$ of unimodular lattices in $\mathbb{R}^k$, choose a norm on $\mathbb{R}^k$ and define the function Δ on X_k by (1.9). Our goal is to prove

Proposition. *There exist positive C_k, C'_k such that*

$$C_k e^{-kz} \geq \Phi_\Delta(z) \geq C_k e^{-kz} - C'_k e^{-2kz} \quad \textit{for all } z \geq 0\,, \tag{7.1}$$

in particular, Δ is k-DL.

The main tool here is the reduction theory for $SL_k(\mathbb{R})/SL_k(\mathbb{Z})$, in particular, a generalization of Siegel's [Si] summation formula. Recall that a vector $\mathbf{v}$ in a lattice $\Lambda \subset \mathbb{R}^k$ is called *primitive* (in Λ) if it is not a multiple of another element of Λ; equivalently, if there exists a basis $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ of Λ with $\mathbf{v}_1 = \mathbf{v}$. Denote by $P(\Lambda)$ the set of all primitive vectors in Λ. Now, given a function φ on $\mathbb{R}^k$, define a function $\hat{\varphi}$ on X_k by $\hat{\varphi}(\Lambda) \stackrel{\text{def}}{=} \sum_{\mathbf{v}\in P(\Lambda)} \varphi(\mathbf{v})$. The following is one of the results of the paper [Si]:

7.2. Theorem. *For any $\varphi \in L^1(\mathbb{R}^k)$, one has $\int_{X_k} \hat{\varphi}\, d\mu = c_k \int_{\mathbb{R}^k} \varphi\, d\mathbf{v}$, where $c_k = \frac{1}{\zeta(k)}$.*

The theorem below is a direct generalization of Siegel's result. For $1 \leq d < k$, say that an ordered d-tuple $(\mathbf{v}_1,\dots,\mathbf{v}_d)$ of vectors in a lattice $\Lambda \subset \mathbb{R}^k$ is *primitive* if it is extendable to a basis of Λ, and denote by $P^d(\Lambda)$ the set of all such d-tuples. Now, given a function φ on $\mathbb{R}^{kd}$, define a function $\hat{\varphi}^d$ on X_k by $\hat{\varphi}^d(\Lambda) \stackrel{\text{def}}{=} \sum_{(\mathbf{v}_1,\dots,\mathbf{v}_d)\in P^d(\Lambda)} \varphi(\mathbf{v}_1,\dots,\mathbf{v}_d)$. Then one has

7.3. Theorem. *For $1 \leq d < k$ and $\varphi \in L^1(\mathbb{R}^{kd})$,*

$$\int_{X_k} \hat{\varphi}^d\, d\mu = c_{k,d} \int_{\mathbb{R}^{kd}} \varphi\, d\mathbf{v}_1 \dots d\mathbf{v}_d\,, \tag{7.2}$$

where $c_{k,d} = \frac{1}{\zeta(k)\cdot\ldots\cdot\zeta(k-d+1)}$.

Sketch of proof. We essentially follow S. Lang's presentation (Yale University lecture course, Spring 1996) of Siegel's original proof. Fix a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ of $\mathbb{R}^k$, denote by G' (resp. Γ') the stabilizer of the ordered d-tuple $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ in G (resp. in Γ'). Then G/G', as a G-homogeneous space, can be naturally identified with an open dense subset of $\mathbb{R}^{kd}$, namely, with the set of linearly independent d-tuples. Similarly Γ/Γ' can be identified with the Γ'-orbit of $(\mathbf{e}_1, \ldots, \mathbf{e}_k)$, which is exactly the set $P^d(\mathbb{Z}^k)$ of primitive d-tuples in $\mathbb{Z}^k$. These identifications allow one to transport the Lebesgue measure from $\mathbb{R}^{kd}$ to a Haar measure $\mu_{G/G'}$ on G/G', and to interpret the summation over $P^d(\mathbb{Z}^k)$ as the integration over the counting measure $\mu_{\Gamma/\Gamma'}$ on Γ/Γ'.

The choice of the normalized Haar measure μ on X_k (and hence of the measures μ_G on G and $\mu_{G/\Gamma'}$ on G/Γ'), together with the aforementioned choice of $\mu_{G/G'}$, uniquely determines the Haar measures $\mu_{G'}$ and $\mu_{G'/\Gamma'}$ on G' and G'/Γ' (note that Γ' is a lattice in G') such that for any $\varphi \in L^1(G/\Gamma')$ one has

$$(7.3)\qquad \int_{X_k}\int_{\Gamma/\Gamma'} \varphi\, d\mu_{\Gamma/\Gamma'}\, d\mu = \int_{G/\Gamma'} \varphi\, d\mu_{G/\Gamma'} = \int_{G/G'}\int_{G'/\Gamma'} \varphi\, d\mu_{G'/\Gamma'}\, d\mu_{G/G'}\,.$$

It remains to take any $\varphi \in L^1(\mathbb{R}^{kd}) \cong L^1(G/G')$, extend it to an integrable function on G/Γ', and notice that the left hand side of (7.2) coincides with that of (7.3), whereas the right hand side of (7.3) can be rewritten as $\mu_{G'/\Gamma'}(G'/\Gamma')\cdot\int_{G/G'}\varphi\, d\mu_{G/G'}$, which is exactly the right hand side of (7.2) with $c_{k,d} = \mu_{G'/\Gamma'}(G'/\Gamma')$. The computation of the exact value of $c_{k,d}$ is not needed for our purposes and is left as an exercise for the reader. $\square$

7.4. *Proof of Proposition 7.1.* Take $z \geq 0$, denote by B the ball in $\mathbb{R}^k$ of radius e^{-z} centered at the origin, and by φ the characteristic function of B. Note that

$$\Delta(\Lambda) \geq z \quad\Rightarrow\quad \log\left(\tfrac{1}{\|\mathbf{v}\|}\right) \geq z \text{ for some } \mathbf{v} \in \Lambda \smallsetminus \{0\} \quad\Rightarrow\quad \Lambda \cap B \neq \{0\}\,,$$

and the latter condition clearly implies that B contains at least two primitive vectors ($\mathbf{v}$ and $-\mathbf{v}$) of Λ. Since $\hat{\varphi}(\Lambda) = \#\big(P(\Lambda)\cap B\big)$, one has

$$(7.5)\qquad \int_{X_k} \hat{\varphi}\, d\mu = \int_{\{\Lambda \mid \Delta(\Lambda)\geq z\}} \hat{\varphi}\, d\mu \geq 2\mu\big(\{\Lambda \mid \Delta(\Lambda) \geq z\}\big)\,.$$

The left hand side, in view of Theorem 7.2, is equal to $c_k\int_{\mathbb{R}^k}\varphi\, d\mathbf{v} = c_k\nu_k e^{-kz}$ (here ν_k is the volume of the unit ball in $\mathbb{R}^k$), hence the upper estimate for $\Phi_\Delta(z)$ in (7.1), with $C_k = \frac{1}{2}c_k\nu_k$.

For the lower estimate, we will demonstrate that lattices Λ with $\hat{\varphi}(\Lambda) > 2$ contribute very insignificantly to the integral in the left hand side of (7.5). Indeed, a standard argument from reduction theory shows that whenever there exist at least two linearly independent vectors in $\Lambda\cap B$, for any $\mathbf{v}_1 \in P(\Lambda)$ one can find $\mathbf{v}_2 \in \Lambda\cap B$ such that $(\mathbf{v}_1, \mathbf{v}_2)$, as well as $(\mathbf{v}_1, -\mathbf{v}_2)$, belongs to $P^2(\Lambda)$. Consequently, one has

$$\hat{\varphi}(\Lambda) = \#\big(P(\Lambda)\cap B\big) \leq \frac{1}{2}\#\big(P^2(\Lambda)\cap(B\times B)\big)$$

whenever $\hat{\varphi}(\Lambda) > 2$. Note that the right hand side is equal to $\frac{1}{2}\overset{\wedge 2}{\psi}(\Lambda)$, where ψ is the characteristic function of $B\times B$ in $\mathbb{R}^{2k}$. Therefore

$$\begin{aligned}\int_{X_k}\hat{\varphi}\, d\mu &= \int_{\{\Lambda\mid\hat{\varphi}(\Lambda)=2\}}\hat{\varphi}\, d\mu + \int_{\{\Lambda\mid\hat{\varphi}(\Lambda)>2\}}\hat{\varphi}\, d\mu\\ &\leq 2\mu\big(\{\Lambda\mid\hat{\varphi}(\Lambda)=2\}\big) + \frac{1}{2}\int_{\{\Lambda\mid\hat{\varphi}(\Lambda)>2\}}\overset{\wedge 2}{\psi}\, d\mu \leq 2\mu\big(\{\Lambda\mid\Delta(\Lambda)\geq z\}\big) + \frac{1}{2}\int_{X_k}\overset{\wedge 2}{\psi}\, d\mu\,.\end{aligned}$$

From Theorems 7.2 and 7.3 it then follows that $2\Phi_\Delta(z) \ge c_k\nu_k e^{-kz} - \frac{1}{2}c_{k,2}(\nu_k)^2 e^{-2kz}$, which finishes the proof of the proposition. $\square$

§8. The Khinchin-Groshev Theorem

8.1. We begin by introducing some terminology. Let $\psi : \mathbb{N} \mapsto (0,\infty)$ be a positive function. Fix $m,n \in \mathbb{N}$ and say that a matrix $A \in M_{m,n}(\mathbb{R})$ (viewed as a system of m linear forms in n variables) is *ψ-approximable*[7] if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that (1.1) holds. Then one can restate Theorem 1.1 as follows:

Theorem. *Let m,n be positive integers and $\psi : [1,\infty) \mapsto (0,\infty)$ a non-increasing continuous function. Then almost every (resp. almost no) $\Lambda \in X_{m+n}$ is (ψ, n)-approximable, provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).*

To prepare for the reduction of this theorem to Theorem 1.7, let us present an equivalent formulation. For a vector $\mathbf{v} \in \mathbb{R}^{m+n}$, denote by $\mathbf{v}^{(m)}$ (resp. $\mathbf{v}_{(n)}$) the vector consisting of first m (resp. last n) components of $\mathbf{v}$. Now, to a matrix $A \in M_{m,n}(\mathbb{R})$ we associate a lattice Λ_A in $\mathbb{R}^{m+n}$ defined by $\Lambda_A \stackrel{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n}$; in other words, $\Lambda_A = \left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{p} \\ \mathbf{q} \end{pmatrix} \middle| \, \mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \right\}$. Clearly $A \in M_{m,n}(\mathbb{R})$ is ψ-approximable iff there exist $\mathbf{v} \in \Lambda_A$ with arbitrarily large $\|\mathbf{v}_{(n)}\|$ such that

$$\|\mathbf{v}^{(m)}\|^m \le \psi(\|\mathbf{v}_{(n)}\|^n)\,. \tag{8.1}$$

Let us say that a lattice $\Lambda \in X_{m+n}$ is *(ψ, n)-approximable* iff there exist $\mathbf{v} \in \Lambda_A$ with arbitrarily large $\|\mathbf{v}_{(n)}\|$ such that (8.1) holds. Now the above theorem can be restated as follows:

- *Let m,n be positive integers and $\psi : [1,\infty) \mapsto (0,\infty)$ a non-increasing continuous function. Then almost every (resp. almost no) lattice of the form Λ_A, $A \in M_{m,n}(\mathbb{R})$, is (ψ, n)-approximable, provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).*

We will see later that the same phenomenon takes place for generic lattices in $\mathbb{R}^{m+n}$. More precisely, we will prove

8.2. Theorem. *Let ψ, m and n be as in Theorem 8.1. Then almost every (resp. almost no) $\Lambda \in X_{m+n}$ is (ψ, n)-approximable, provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).*

In fact it is not a priori clear how to derive Theorem 8.2 from Theorem 1.1 and vice versa. We will do it by restating these theorems in the language of flows on the space of lattices. For that we first need a change of variables technique formalized in the following

8.3. Lemma. *Fix $m,n \in \mathbb{N}$ and $x_0 > 0$, and let $\psi : [x_0,\infty) \mapsto (0,\infty)$ be a non-increasing continuous function. Then there exists a unique continuous function $r : [t_0,\infty) \mapsto \mathbb{R}$, where $t_0 = \frac{m}{m+n}\log x_0 - \frac{n}{m+n}\log\psi(x_0)$, such that*

$$\text{the function} \quad \lambda(t) \stackrel{\text{def}}{=} t - nr(t) \quad \text{is strictly increasing and tends to } \infty \text{ as } t \to +\infty\,, \tag{8.2a}$$

$$\text{the function} \quad L(t) \stackrel{\text{def}}{=} t + mr(t) \quad \text{is nondecreasing}\,, \tag{8.2b}$$

[7] The authors are grateful to M. Dodson for a permission to modify his terminology introduced in [Do].

and

$$\psi(e^{t-nr(t)}) = e^{-t-mr(t)} \quad \forall\, t \ge t_0\,. \tag{8.3}$$

Conversely, given $t_0 \in \mathbb{R}$ *and a continuous function* $r : [t_0, \infty) \mapsto \mathbb{R}$ *such that* (8.2ab) *hold, there exists a unique continuous non-increasing function* $\psi : [x_0, \infty) \mapsto (0, \infty)$*, with* $x_0 = e^{t_0 - nr(t_0)}$*, satisfying* (8.3)*. Furthermore, for a nonnegative integer* q*,*

$$I_1 \overset{\text{def}}{=} \int_{x_0}^{\infty} (\log x)^q \psi(x)\, dx < \infty \quad \textit{iff} \quad I_2 \overset{\text{def}}{=} \int_{t_0}^{\infty} t^q e^{-(m+n)r(t)}\, dt < \infty\,.$$

Proof. The claimed correspondence becomes transparent if one uses the variables $L = -\log\psi$, $\lambda = \log x$, and the function $P(\lambda) \overset{\text{def}}{=} -\log\psi(e^\lambda)$. Given $t \ge t_0$, one can define $\big(\lambda(t), L(t)\big)$ to be the unique intersection point of the graph of the nondecreasing function $L = P(\lambda)$ and the decreasing straight line $L = \frac{m+n}{n}t - \frac{m}{n}\lambda$, and then put

$$r(t) = \frac{L(t) - \lambda(t)}{m+n}\,. \tag{8.4}$$

The properties (8.2ab) and (8.3) are then straightforward. Conversely, given the function $r(\cdot)$ with (8.2ab) and $\lambda \ge \lambda_0 \overset{\text{def}}{=} t_0 - nr(t_0)$, one defines $P(\lambda)$ to be equal to $L\big(t(\lambda)\big)$, where $L(\cdot)$ is as in (8.2b) and $t(\cdot)$ is the function inverse to $\lambda(\cdot)$ of (8.2a).

Further, the integral I_1 is equal to $\int_{\lambda_0}^\infty \lambda^q e^{\lambda - P(\lambda)}\, d\lambda$, while I_2, in view of (8.2ab) and (8.4), can be written as $\int_{\lambda_0}^\infty \big(\frac{m}{m+n}\lambda + \frac{n}{m+n}P(\lambda)\big)^q e^{\lambda - P(\lambda)} \big(\frac{m}{m+n}d\lambda + \frac{n}{m+n}dP(\lambda)\big) \ge I_1$. It remains to assume $I_1 < \infty$ and prove that the following integrals are finite:

$$I_3 = \int_{\lambda_0}^\infty \lambda^q e^{\lambda - P(\lambda)}\, dP(\lambda), \quad I_4 = \int_{\lambda_0}^\infty P(\lambda)^q e^{\lambda - P(\lambda)}\, d\lambda, \quad I_5 = \int_{\lambda_0}^\infty P(\lambda)^q e^{\lambda - P(\lambda)}\, dP(\lambda)\,.$$

Integration by parts reduces I_3 to the form

$$I_3 = -\int_{\lambda_0}^\infty \lambda^q e^\lambda\, d\big(e^{-P(\lambda)}\big) = -\, \lambda^q e^{\lambda - P(\lambda)}\Big|_{\lambda_0}^\infty + \int_{\lambda_0}^\infty e^\lambda(\lambda^q + q\lambda^{q-1})e^{\lambda - P(\lambda)}\, d\lambda\,,$$

where both terms are finite due to the finiteness of I_1. To estimate I_4, one writes

$$I_4 = \int_{\lambda \ge \lambda_0,\, P(\lambda) < 2\lambda} P(\lambda)^q e^{\lambda - P(\lambda)}\, d\lambda + \int_{\lambda \ge \lambda_0,\, P(\lambda) \ge 2\lambda} P(\lambda)^q e^{\lambda - P(\lambda)}\, d\lambda\,;$$

the first term is clearly bounded from above by $2^q I_1$, while the integrand in the second term is for large enough values of λ not greater than $2^q\lambda^q e^{-\lambda}$. This implies that I_4 is also finite. Finally,

$$\begin{aligned} I_5 &= \int_{\lambda \ge \lambda_0,\, P(\lambda) < 2\lambda} P(\lambda)^q e^{\lambda - P(\lambda)}\, dP(\lambda) + \int_{\lambda \ge \lambda_0,\, P(\lambda) \ge 2\lambda} P(\lambda)^q e^{\lambda - P(\lambda)}\, dP(\lambda) \\ &\le 2^q I_3 + \int_{\lambda_0}^\infty P(\lambda)^q e^{-P(\lambda)/2}\, dP(\lambda) < \infty\,, \end{aligned}$$

which finishes the proof of the lemma. ☐

In what follows, we will denote by $\mathcal{D}_{m,n}(\psi)$ (after S.G. Dani) the function r corresponding to ψ by the above lemma. Note also that r does not have to be monotonic, but is always quasi-increasing (as defined in §2.4) in view of (8.2b).

8.4. Example. The easiest special case is given by $\psi(x) = \varepsilon/x$ for a positive constant ε. Then the equation (8.3) gives $r(t) = \frac{1}{m+n}\log(\frac{1}{\varepsilon})$, so the correspondence $\mathcal{D}_{m,n}$ sends such a function ψ to a constant. Recall that $A \in M_{m,n}(\mathbb{R})$ is said to be *badly approximable* if it is not $\frac{\varepsilon}{x}$-approximable for some $\varepsilon > 0$. In [D], Dani proved that A is badly approximable iff the trajectory $\{f_t\Lambda_A \mid t \geq 0\}$, with f_t as in (1.10), is bounded in X_{m+n}. Note that in view of Mahler's Compactness Criterion (see [R1, Corollary 10.9]), the latter condition is equivalent to the existence of an upper bound for $\{\Delta(f_t\Lambda_A) \mid t \geq 0\}$, with Δ as in (1.9).

8.5. We are now going to prove a generalization of the aforementioned result of Dani.

Theorem. *Let ψ, m and n be as in Theorem 8.1, Δ as in* (1.9), *$\{f_t\}$ as in* (1.10) *Then $\Lambda \in X_{m+n}$ is (ψ, n)-approximable iff there exist arbitrarily large positive t such that*

$$\Delta(f_t\Lambda) \geq r(t)\,, \tag{8.5}$$

where $r = \mathcal{D}_{m,n}(\psi)$. In particular, $A \in M_{m,n}(\mathbb{R})$ is ψ-approximable iff there exist arbitrarily large positive t such that

$$\Delta(f_t\Lambda_A) \geq r(t)\,. \tag{8.5A}$$

Proof. Assume that (8.1) holds for some $\mathbf{v} \in \Lambda$, and note that, by definition of f_t and Δ, to prove (8.5) it suffices to find t such that

$$e^{t/m}\|\mathbf{v}^{(m)}\| \leq e^{-r(t)} \tag{8.6a}$$

and

$$e^{-t/n}\|\mathbf{v}_{(n)}\| \leq e^{-r(t)} \tag{8.6b}$$

Now define t by

$$\|\mathbf{v}_{(n)}\|^n = e^{t-nr(t)}\,. \tag{8.7}$$

In view of (8.2a), one can do this whenever $\|\mathbf{v}_{(n)}\|$ is large enough. Then (8.6b) follows immediately, and one can write

$$\|\mathbf{v}^{(m)}\|^m \underset{(8.1)}{\leq} \psi(\|\mathbf{v}_{(n)}\|^n) \underset{(8.7)}{=} \psi(e^{t-nr(t)}) \underset{(8.3)}{=} e^{-t-mr(t)}\,,$$

which readily implies (8.6a). Lastly, again in view of (8.2a), t will be arbitrarily large if one chooses $\|\mathbf{v}_{(n)}\|$ arbitrarily large as well.

For the converse, let us first take care of the case when

$$\mathbf{v}^{(m)} = 0 \text{ for some } \mathbf{v} \in \Lambda \smallsetminus \{0\}\,. \tag{8.8}$$

Then one can take integral multiples of this $\mathbf{v}$ to produce infinitely many vectors satisfying (8.1); thus lattices with (8.8) are (ψ, n)-approximable for any function ψ. Otherwise, assume that (8.5) holds for a sufficiently large t. This immediately gives a vector $\mathbf{v} \in \Lambda$ satisfying (8.6a) and (8.6b), and one can write

$$\|\mathbf{v}^{(m)}\|^m \underset{(8.6a)}{\leq} e^{-t-mr(t)} \underset{(8.3)}{=} \psi(e^{t-nr(t)}) \underset{(8.6b) \text{ and the monotonicity of } \psi}{\leq} \psi(\|\mathbf{v}_{(n)}\|^n)\,.$$

Finally, if t is taken arbitrarily large, $\|\mathbf{v}^{(m)}\|$ becomes arbitrarily small in view of (8.6a), and yet can not equal zero, so $\|\mathbf{v}_{(n)}\|$ must be arbitrarily large by the discreteness of Λ. $\square$

8.6. *Proof of Theorem 8.2.* In view of the above theorem and Lemma 8.3, it suffices to prove the following

Theorem. *Given $m,n \in \mathbb{N}$, Δ as in* (1.9), $\{f_t\}$ *as in* (1.10) *and a continuous quasi-increasing function $r : [t_0,\infty) \mapsto \mathbb{R}$, for almost every (resp. almost no) $\Lambda \in X_{m+n}$ there exist arbitrarily large positive t such that* (8.5) *holds, provided the integral $\int_{t_0}^\infty e^{-(m+n)r(t)}\,dt$ diverges (resp. converges).*

Proof. From Corollary 2.4 and Lemma 2.8 it is clear that the above statement is a straightforward consequence of the family $\mathcal{B}(\Delta)$ being Borel-Cantelli for f_1. The latter, in its turn, immediately follows from Theorem 1.7 and Proposition 7.1. □

8.7. *Proof of Theorem 1.1.* Similarly, Theorem 1.1 follows from

Theorem. *Given $m,n \in \mathbb{N}$, Δ as in* (1.9), $\{f_t\}$ *as in* (1.10) *and a continuous quasi-increasing function $r : [t_0,\infty) \mapsto \mathbb{R}$, for almost every (resp. almost no) $A \in M_{m,n}(\mathbb{R})$ there exist arbitrarily large positive t such that* (8.5A) *holds, provided the integral $\int_{t_0}^\infty e^{-(m+n)r(t)}\,dt$ diverges (resp. converges).*

Proof. It is easy to see (cf. [D, 2.11]) that any lattice $\Lambda \in X_{m+n}$ can be written in the form

$$\Lambda = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix} \Lambda_A \,,$$

for some $A \in M_{m,n}(\mathbb{R})$, $B_1 \in M_{m,m}(\mathbb{R})$, $B_2 \in M_{n,m}(\mathbb{R})$ and $B_3 \in M_{n,n}(\mathbb{R})$ with $\det(B_1)\det(B_3) = 1$. Therefore one can write

$$f_t\Lambda = f_t \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix} f_{-t} f_t \Lambda_A = \begin{pmatrix} B_1 & 0 \\ e^{-(t/m+t/n)}B_2 & B_3 \end{pmatrix} f_t\Lambda_A \,.$$

From this and the uniform continuity of Δ it follows that for some positive C (dependent on Λ) one has $\sup_{t>0} |\Delta(f_t\Lambda) - \Delta(f_t\Lambda_A)| < C$. If $\int_{t_0}^\infty e^{-(m+n)r(t)}\,dt$ diverges (resp. converges), by Theorem 8.6 the set of lattices Λ such that for any (resp. for some) $C > 0$ there exist arbitrarily large positive t with $\Delta(f_t\Lambda) \ge r(t) + C$ (resp. with $\Delta(f_t\Lambda) \ge r(t) - C$), has full (resp. zero) measure in X_{m+n}. Therefore, by Fubini, the set of $A \in M_{m,n}(\mathbb{R})$ such that (8.5A) holds for arbitrarily large t has full (resp. zero) measure in $M_{m,n}(\mathbb{R})$. □

8.8. Remark. It is also possible to argue in the opposite direction and deduce Theorem 8.6 from Theorem 8.7. (Cf. [D], where the abundance of bounded orbits for certain flows on X_{m+n} was deduced from W. Schmidt's result on badly approximable systems of linear forms, vs. [KM], where ergodic theory was used to construct bounded orbits, thus providing another proof of the aforementioned result of Schmidt.) In other words, one can derive logarithm laws for specific flows on X_{m+n} simply by applying Theorem 8.5 to translate the Khinchin-Groshev Theorem into the dynamical language. As a historical note, the authors want to point out that this is exactly what they understood first and what prompted them to start working on this circle of problems.

9. Multiplicative approximation of lattices

9.1. As a motivation, let us consider the case $m = n = 1$ of Theorem 8.2. The inequality (8.1) then transforms into

(9.1) $$|v_1| \le \psi(|v_2|)\,, \quad \text{or} \quad |v_1||v_2| \le |v_2|\psi(|v_2|)\,,$$

where $\mathbf{v} = (v_1, v_2)$ is a vector from a lattice $\Lambda \in X_2$. Since ψ is bounded, the fact that (9.1) holds for vectors $\mathbf{v}$ with arbitrarily large $|v_2|$ implies that one has $\|\mathbf{v}\| = |v_2|$ for infinitely many $\mathbf{v} \in \Lambda$ satisfying (9.1); therefore (9.1) can be replaced by (1.11). Conversely, if (1.11) holds for infinitely many $\mathbf{v} \in \Lambda$, then either Λ or its mirror reflection around the axis $v_1 = v_2$ is $(\psi, 1)$-approximable. This way one gets an equivalent form of the $m = n = 1$ case of Theorem 8.2 as follows:

- *With ψ as in Theorem 8.1, for almost every (resp. almost no) $\Lambda \in X_2$ there exist infinitely many $\mathbf{v} \in \Lambda$ with* (1.11)*, provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).*

This suggests a natural generalization and (sigh!) another definition: for an integer $k \geq 2$, say that $\Lambda \in X_k$ is *ψ-multiplicatively approximable* (to be abbreviated as ψ-MA) if there exist infinitely many $\mathbf{v} \in \Lambda$ satisfying (1.11). Thus the above theorem can be restated as follows:

- *For ψ as in Theorem 8.1, almost every (resp. almost no) $\Lambda \in X_2$ is ψ-MA, provided the integral $\int_1^\infty \psi(x)\,dx$ diverges (resp. converges).*

A question, raised by M. Skriganov in [Sk, p. 23], amounts to considering a family of functions $\psi_q(x) = 1/x(\log x)^q$ and looking for a critical exponent $q_0 = q_0(k)$ such that almost all (resp. almost no) $\Lambda \in X_k$ are ψ_q-MA if $q \leq q_0$ (resp. if $q > q_0$). It is shown in [Sk] that $q_0(k)$ must be positive and not greater than $k - 1$. In this section we prove Theorem 1.11, which, using the above terminology, reads as follows:

- *Let $\psi : [1, \infty) \mapsto (0, \infty)$ be a non-increasing continuous function and k an integer greater than 1. Then almost every (resp. almost no) $\Lambda \in X_k$ is ψ-MA, provided the integral $\int_1^\infty (\log x)^{k-2}\psi(x)\,dx$ diverges (resp. converges).*

In particular, this proves the existence of $q_0(k)$ and gives its exact value, namely, $q_0(k) = k - 1$.

9.2. In order to reduce Theorem 1.11 to Theorem 1.9, we need an analogue of the correspondence of Theorem 8.5. Again, the special case given by $\psi(x) = \varepsilon/x$ and $r \equiv$ const is worth mentioning. Recall that Λ is called *admissible* (cf. [Sk, p. 6]) if it is not $\frac{\varepsilon}{x}$-MA for some $\varepsilon > 0$. It easily follows from Mahler's Compactness Criterion (and is mentioned in [Sk, p. 14]) that a lattice is admissible iff its orbit under the diagonal subgroup of $SL_k(\mathbb{R})$ is bounded in X_k. To generalize this observation, identify the Lie algebra $\mathfrak{d}$ of traceless diagonal $k \times k$ matrices with $\{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k \mid \sum_{i=1}^k t_i = 0\}$, denote by $f_\mathbf{t}$ the element of $SL_k(\mathbb{R})$ given by

$$f_\mathbf{t} = \exp(\mathbf{t}) = \mathrm{diag}(e^{t_1}, \dots, e^{t_k})\,, \tag{9.2}$$

and let $\|\mathbf{t}\|_- \stackrel{\text{def}}{=} \max\{|t_i| \bigm| t_i \leq 0\}$.

Theorem. *Let ψ be as in Theorem 8.1, k an integer greater than* 1*, Δ as in* (1.9) *and $\{f_\mathbf{t}\}$ as in* (9.2)*. Then $\Lambda \in X_k$ is ψ-MA iff there exist $\mathbf{t} \in \mathfrak{d}$ arbitrarily far from* 0 *such that*

$$\Delta(f_\mathbf{t}\Lambda) \geq r(\|\mathbf{t}\|_-)\,, \tag{9.3}$$

where $r = \mathcal{D}_{k-1,1}(\psi)$.

Proof. Assume that (1.11) holds for some $\mathbf{v} \in \Lambda$; our goal is to find $\mathbf{t}$ such that

$$e^{t_i}|v_i| \leq e^{-r(\|\mathbf{t}\|_-)} \text{ for all } 1 \leq i \leq k\,. \tag{9.4}$$

We will do it in two steps. First define $t \in \mathbb{R}$ by $\|\mathbf{v}\| = e^{t-r(t)}$ (as before, one uses (8.2a) to justify this step if $\|\mathbf{v}\|$ is large enough). Note that in view of (8.3) one then has

$$\psi(\|\mathbf{v}\|) = \psi(e^{t-r(t)}) = e^{-t-(k-1)r(t)}\,.$$

To define $\mathbf{t}$, assume without loss of generality that $|v_i| \ge |v_{i+1}|$ for all $i < k$, and put $e^{t_1} = \frac{e^{-r(t)}}{|v_1|} = \frac{e^{-r(t)}}{\|\mathbf{v}\|} = e^{-t}$, and then, inductively, $e^{t_i} = \min\left(\frac{e^{-r(t)}}{|v_i|}, e^{-(t_1+\dots+t_{i-1})}\right)$. Then one can check that:

- e^{t_i} is not greater than $\frac{e^{-r(t)}}{|v_i|}$ for all i,
- $\sum_{i=1}^k t_i = 0$, and
- $t = -t_1 = -\min_{1\le i\le k} t_i = \|\mathbf{t}\|_-$.

Therefore (9.4) is satisfied, and it remains to observe that, again in view of (8.2a), $\|\mathbf{t}\|_-$ will be arbitrarily large if one chooses $\|\mathbf{v}\|$ arbitrarily large as well.

For the converse, we have to first take care of the case when

(9.5) $$\mathbf{v}_i = 0 \text{ for some } \mathbf{v} \in \Lambda \smallsetminus \{0\} \text{ and } 1 \le i \le k$$

(in [Sk] such lattices are called *not weakly admissible*). Clearly one can take integral multiples of this $\mathbf{v}$ to produce infinitely many vectors satisfying (1.11); thus lattices with (9.5) are ψ-MA for any function ψ. Otherwise, assume that (9.3) holds for some $\mathbf{t} \in \mathfrak{d}$ with sufficiently large $\|\mathbf{t}\|_-$. This immediately gives a vector $\mathbf{v} \in \Lambda$ satisfying (9.4). Let us again order the components of $\mathbf{v}$ so that $|v_1| \ge \dots \ge |v_k|$. Note that without loss of generality one can assume that $\|\mathbf{t}\|_- = -t_1$ (otherwise, if $\|\mathbf{t}\|_- = -t_j > -t_1$, one can interchange t_1 and t_j without any damage to (9.4)). Now one can multiply the inequalities (9.4) for $i = 2, \dots, n$ by each other to get $\prod_{2\le i\le k} e^{t_i}|v_i| \le e^{-(k-1)r(\|\mathbf{t}\|_-)}$, which makes $\Pi(\mathbf{v})/\|\mathbf{v}\|$ to be not greater than

$$e^{t_1-(k-1)r(\|\mathbf{t}\|_-)} = e^{-\|\mathbf{t}\|_- -(k-1)r(\|\mathbf{t}\|_-)} \underset{(8.3)}{=} \psi(e^{\|\mathbf{t}\|_- - r(\|\mathbf{t}\|_-)}) \underset{\text{(9.4) and the monotonicity of }\psi}{\le} \psi(\|\mathbf{v}\|)$$

as desired. Finally, recall that $\mathbf{t}$ can be taken arbitrarily far from 0. Let i be such that $t_i = \max_{1\le j\le k} t_j$. Then (9.4) makes $|v_i|$ arbitrarily small and yet positive, so $\|\mathbf{v}\|$ must be arbitrarily large by the discreteness of Λ. $\square$

9.3. *Proof of Theorem 1.11.* In view of the correspondence described in the above theorem, we have to prove the following

- *Given an integer $k > 1$, Δ as in* (1.9)*, $\mathfrak{d}$ as in §9.2, $\{f_{\mathbf{t}}\}$ as in* (9.2) *and a continuous quasi-increasing function $r : [t_0, \infty) \mapsto \mathbb{R}$, for almost every (resp. almost no) $\Lambda \in X_k$ there exist $\mathbf{t} \in \mathfrak{d}$ arbitrarily far from* 0 *such that* (9.3) *holds, provided the integral $\int_{t_0}^{\infty} t^{k-2}e^{-kr(t)}\,dt$ diverges (resp. converges).*

Note that the function $\mathbf{t} \mapsto \|\mathbf{t}\|_-$ becomes a norm when restricted to any Weyl chamber of $\mathfrak{d}$. Therefore one can decompose $\mathfrak{d}$ as a union of such chambers $\mathfrak{d}_j$ and then apply Theorem 1.9, powered by Proposition 7.1, to conclude that the family $\mathcal{B}(\Delta)$ is Borel-Cantelli for $\{f_{\mathbf{t}}\}$, where $\mathbf{t}$ runs through the intersection of $\mathfrak{d}_j$ with an arbitrary lattice in $\mathfrak{d}$. The statement of the theorem then immediately follows from Corollary 2.4 and Lemma 2.8. $\square$

§10. Concluding remarks and open questions

10.1. It seems natural to conjecture that the conclusion of Theorem 1.12 (isolation properties of the restriction of ρ_0 to any simple factor of G), and hence of Corollary 3.5 (exponential decay of correlation coefficients of smooth functions), are satisfied for uniform lattices $\Gamma \subset G$ as well. This is clearly the case when all factors of G have property (T); otherwise the problem stands open.

10.2. In view of the result of W. Philipp mentioned in §1.5 (or a similar result for expanding rational maps of Julia sets announced recently by R. Hill and S. Velani), it seems natural to ask whether the family of all balls in G/Γ will be Borel-Cantelli for an element $\exp(\mathbf{z})$ of G as in Theorem 1.7. For fixed $x_0 \in G/\Gamma$, this would measure the rate with which a typical orbit approaches x_0, in particular, in the form of a logarithm law for the function $\Delta(x) = \log\left(\frac{1}{\mathrm{dist}(x_0,x)}\right)$. This function satisfies (k-DL) with $k = \dim(G/\Gamma)$, but is not uniformly continuous, therefore super-level sets of Δ cannot be adequately approximated by smooth functions.

On the other hand, D. Dolgopyat [Dol] recently proved a number of limit theorems for partially hyperbolic dynamical systems. In particular he showed that if f is a partially hyperbolic diffeomorphism of a compact Riemannian manifold X, then the family of all balls in X is Borel-Cantelli for f, provided a certain additional assumption (involving rate of convergence of averages along pieces of unstable leaves) is satisfied. Using [KM, Propositions 2.4.8 or A.6] this assumption can be checked when G, Γ and $f = \exp(\mathbf{z})$ are as in Theorem 1.7, $X = G/\Gamma$ is compact and all simple factors of G have property (T). See also [CK, CR] for other results in this direction.

10.3. We now roughly sketch modifications one has to make in order to consider flows on reducible homogeneous spaces. If G is a connected semisimple center-free Lie group without compact factors and Γ is a lattice in G, one can find connected normal subgroups $G_1, \dots, G_l$ of G such that $G = \prod_{i=1}^l G_i$ (direct product), $\Gamma_i \stackrel{\mathrm{def}}{=} G_i \cap \Gamma$ is an irreducible lattice in G_i for each i, and $\prod_{i=1}^l \Gamma_i$ has finite index in Γ (cf. [R1, Theorem 5.22]). As a consequence of the above, G/Γ is finitely covered by the direct product of the spaces G_i/Γ_i. Denote by p_i the projection from G onto G_i. Then one can apply Corollary 3.5 to the factors G_i/Γ_i (more precisely, to the noncompact ones) and deduce that Theorem 4.3 (hence Theorem 1.8 as well) holds in this generality provided the condition (ED) is replaced by

$$p_i(F) \text{ is ED for all } i = 1, \dots, l\,. \tag{10.1}$$

Similarly one takes care of the case when G has a nontrivial center: then G can be written as an almost direct product of the groups G_i, and the maps p_i are defined to be the projections $G \mapsto G/\prod_{j\neq i} G_j$.

Specializing to the case $F = \{\exp(t\mathbf{z}) \mid t \in \mathbb{N}\}$, with $\mathbf{z} \in \mathfrak{a}$ as in Theorem 1.7, it is easy to see that (10.1) is satisfied whenever $p_i(\mathbf{z})$ is nontrivial for all i (here with some abuse of notation we let p_i be the projections of the corresponding Lie algebras). The latter condition holds for a generic element $\mathbf{z} \in \mathfrak{a}$. Furthermore, one can prove that the k-DL property of the distance function can be lifted to the direct product of metric spaces. (More precisely, if Δ_i is a k_i-DL function on X_i, $1 \le i \le l$, then $\sqrt{\Delta_1^2 + \cdots + \Delta_l^2}$ is $(\min_{1\le i\le l} k_i)$-DL function on $\prod_{i=1}^l X_i$.) Therefore one can argue as in §6 and prove Theorem 1.4 without assuming that the space Y is irreducible.

10.4. Suppose that G, Γ and $F = \{f_t\}$ are as in Theorem 1.8, and let Δ be a uniformly continuous function on G/Γ such that

$$\forall\, c < 1\ \exists\, \delta > 0 \text{ such that } \Phi_\Delta(z+\delta) \ge c \cdot \Phi_\Delta(z) \text{ for large enough } z\,. \tag{10.2}$$

For such functions one can prove a refinement of Theorem 4.3: if $\{r_t\}$ is a sequence of real numbers satisfying (4.5), then for almost all $x \in G/\Gamma$ one has

$$\lim_{N\to\infty} \frac{\#\{1 \le t \le N \mid \Delta(f_t x) \ge r_t\}}{\sum_{s=1}^N \Phi_\Delta\big(r_t\big)} = 1\,.$$

It is easy to see that (7.1) implies (10.2), therefore such a refinement holds for the function Δ on $SL_k(\mathbb{R})/SL_k(\mathbb{Z})$ given by (1.9). It seems very likely that distance functions on locally symmetric spaces satisfy (10.2) as well; in other words, one can write exact asymptotics for the measure of the complement of a ball of radius z, not only bound it from both sides by const$\cdot e^{-kz}$. However, the proof is beyond our reach, since in order to use the main tools of our proof (reduction theory and the quasi-isometry with a Siegel set) one has to sacrifice a multiplicative constant.

Appendix

A.0. Let ρ be a unitary representation of a locally compact second countable group G in a separable Hilbert space V. Say that a sequence $\{v_t \mid t \in \mathbb{N}\} \subset V$ is *asymptotically ρ-invariant* if $v_t \neq 0$ for all sufficiently large t, and $\|\rho(g)v_t - v_t\|/\|v_t\| \to 0$ as $t \to \infty$ uniformly on compact subsets of G. Then ρ is isolated from I_G in the Fell topology iff there are no asymptotically ρ-invariant sequences $\{v_t\} \subset V$.

Let now (X, μ) be a probability space, and $(g, x) \mapsto gx$ a μ-preserving action of G on X. Denote by $L^2_0(X, \mu)$ the subspace of $L^2(X, \mu)$ orthogonal to constant functions, and by ρ_0 the regular representation of G on $L^2_0(X, \mu)$. Now, with some abuse of terminology, say that a sequence $\{A_t \mid t \in \mathbb{N}\}$ of nontrivial measurable subsets of X is *asymptotically invariant* if the sequence of functions $1_{A_t} - \mu(A_t)$ is asymptotically ρ_0-invariant. Equivalently, if

$$\text{(AI)} \qquad \mu(A_t \triangle g A_t)/\mu(A_t) \to 0 \text{ as } t \to \infty \text{ uniformly on compact subsets of } G\,.$$

Further, we will say that $\{A_t\}$ is a *0-sequence* if $\lim_{t\to\infty} \mu(A_t) = 0$.

Now we can state the following useful criterion for ρ_0 being close to I_G:

Proposition. *Let G be a locally compact second countable group acting ergodically on a probability space (X, μ). Then the following two conditions are equivalent:*

(i) *there exists an asymptotically invariant 0-sequence of subsets A_t of X;*
(ii) *ρ_0 is not isolated from I_G.*

The implication (i)$\Rightarrow$(ii) is clear: by definition, the sequence of functions $1_{A_t} - \mu(A_t)$ is asymptotically ρ_0-invariant whenever $\{A_t\}$ is asymptotically invariant. K. Schmidt [S], using a result of J. Rosenblatt [Ro], proved the converse for countable groups G; in fact, he showed that both conditions are equivalent to

(iii) G has more than one invariant mean on $L^\infty(X, \mu)$.

In [FS], A. Furman and Y. Shalom extended the approach of Rosenblatt and Schmidt to uncountable groups. In particular, assuming G is locally compact, they proved the implication (ii)$\Rightarrow$(iii), of which the converse is in this generality not always true. Our proof of Proposition A.0 is based on the ideas of Rosenblatt-Schmidt-Furman-Shalom. However we have chosen to completely avoid any use of invariant means, in the hope to make the argument more transparent and less involved.

Proof of Proposition A.0. Suppose we are given a sequence of functions $\{\varphi_t\} \in L^2_0(X, \mu)$ which is asymptotically ρ_0-invariant. Without loss of generality we can assume that all the functions φ_t have L^2-norm 1. Note also that any weak limit point of the sequence $\{\varphi_t\}$ must be ρ_0-invariant, hence (by the ergodicity of the G-action on X) equal to zero. Thus, by choosing a subsequence, we can assume that $\varphi_t \to 0$ weakly as $t \to \infty$.

Our goal is to produce an asymptotically invariant 0-sequence $\{A_t\}$ of subsets of X. Define a sequence $\{\sigma_t\}$ of probability measures on $\mathbb{R}$ by

$$\sigma_t(A) = \mu\big(\varphi^{-1}(A)\big)\,, \quad A \subset \mathbb{R}\,.$$

Observe that

$$\int_{\mathbb{R}} z\, d\sigma_t(z) = 0 \text{ and } \int_{\mathbb{R}} z^2\, d\sigma_t(z) = 1\,. \tag{A.0}$$

In view of the last equality, we may assume that σ_t converges weakly on compacta to a probability measure σ on $\mathbb{R}$. The construction of the desired sequence of sets will crucially depend on this measure. Following [S] and [FS], we consider two cases.

Case 1. *The limit measure is concentrated on one point $a \in \mathbb{R}$.*

A.1.1. Let us, following [FS], first show that $a = 0$. Indeed, using (A.0), for any $t \in \mathbb{N}$ and $N > 0$ one can write

$$\left|\int_{-N}^{N} z\, d\sigma_t(z)\right| = \left|\int_{|z|>N} z\, d\sigma_t(z)\right| = \frac{1}{N}\left|N\int_{|z|>N} z\, d\sigma_t(z)\right| \le \frac{1}{N}\left|\int z^2\, d\sigma_t(z)\right| = \frac{1}{N}\,.$$

Choosing N large enough and σ_t close enough to σ, one deduces that $|a| = \left|\int_{-N}^{N} z\, d\sigma(z)\right|$ must be very small, which is only possible if $a = 0$. In particular, this implies that for any $C > 0$,

$$\int\limits_{\{|\varphi_t|<C\}} \varphi_t^2\, d\mu = \int_{-C}^{C} z^2\, d\sigma_t(z) \to \int_{-C}^{C} z^2\, d\sigma(z) = 0\,. \tag{A.1.1}$$

A.1.2. The next step is to pass from functions $\{\varphi_t\}$ with zero mean value to another sequence $\{h_t\}$ of nonnegative integrable functions. Namely we define

$$h_t(x) = \begin{cases} \varphi_t^2(x), & |\varphi_t(x)| \ge 1 \\ 0, & |\varphi_t(x)| < 1 \end{cases} \tag{A.1.2}$$

In what follows, we denote by $\|h\|_1$ the L^1-norm of a function h, and keep the notation $\|\cdot\|$ for the L^2-norm.

Lemma. *As $t \to \infty$, $\|h_t\|_1 \to 1$ and $\|h_t - gh_t\|_1 \to 0$ uniformly on compact subsets of G.*

Proof. Note first that $\|\varphi_t^2\|_1 = \|\varphi_t\|^2 = 1$, while $\|h_t\|_1 - \|\varphi_t^2\|_1 = \int_{\{|\varphi_t(x)|<1\}} \varphi_t^2\, d\mu \to 0$ in view of (A.1.1). Now for any $g \in G$ one can write

$$\|h_t - gh_t\|_1 = \int\limits_{\{|g\varphi_t|<1,\, |\varphi_t|\ge 1\}} \varphi_t^2\, d\mu + \int\limits_{\{|\varphi_t|<1,\, |g\varphi_t|\ge 1\}} g\varphi_t^2\, d\mu + \int\limits_{\{|\varphi_t|\ge 1,\, |g\varphi_t|\ge 1\}} |\varphi_t^2 - g\varphi_t^2|\, d\mu\,.$$

The first integral in the r.h.s. is not greater than

$$\int\limits_{\{1\le|\varphi_t|<2\}} \varphi_t^2\, d\mu \;+ \int\limits_{\{|\varphi_t|\ge 2,\, |g\varphi_t|\le|\varphi_t|/2\}} \varphi_t^2\, d\mu \;\le \int\limits_{\{1\le|\varphi_t|<2\}} \varphi_t^2\, d\mu \;+\; \frac{4}{3}\int\limits_{\{|\varphi_t|\ge 2\}} |\varphi_t^2 - g\varphi_t^2|\, d\mu\,;$$

similarly, $\displaystyle\int_{\{|\varphi_t|<1,\,|g\varphi_t|\geq 1\}} g\varphi_t^2\,d\mu \leq \int_{\{1\leq|g\varphi_t|<2\}} g\varphi_t^2\,d\mu + \frac{4}{3}\int_{\{|g\varphi_t|\geq 2\}} |\varphi_t^2 - g\varphi_t^2|\,d\mu$. Thus, using (A.1.1) and the G-invariance of μ, one gets

$$\limsup_{t\to\infty} \|h_t - gh_t\|_1 \leq \frac{11}{3}\cdot \limsup_{t\to\infty} \|\varphi_t^2 - g\varphi_t^2\|_1\,.$$

But $\|\varphi_t^2 - g\varphi_t^2\|_1 = \|(\varphi_t - g\varphi_t)(\varphi_t + g\varphi_t)\|_1 \leq 2\|\varphi_t - g\varphi_t\|$, and the latter L^2-norms tend to zero uniformly on compact subsets of G, hence the claim. $\square$

A.1.3. The next step of the proof is to pass from functions to sets. Here we use the following trick, dating back to I. Namioka [N]: if h is a nonnegative function on X and $z \geq 0$, denote by $B_{z,h}$ the subset of X given by

$$B_{z,h} \stackrel{\text{def}}{=} \{x \in X \mid h(x) \geq z\}\,.$$

Then one can reconstruct the value of $h(x)$ as the Lebesgue measure of the set $\{z \geq 0 \mid x \in B_{z,h}\}$. Moreover, if $g \in G$, the absolute value of $(gh)(x) - h(x)$ is equal to the measure of $\{z \geq 0 \mid x \in B_{z,h} \triangle B_{z,gh}\}$. Therefore, assuming h is integrable, its L^1-norm is equal to

$$\|h\|_1 = \int_X \int_0^\infty 1_{\{z \mid x\in B_{z,h}\}}\,dz\,d\mu(x) = \int_0^\infty \int_X 1_{\{z \mid x\in B_{z,h}\}}\,d\mu(x)\,dz = \int_0^\infty \mu(B_{z,h})\,dz\,;$$

similarly,

$$\|gh - h\|_1 = \int_X \int_0^\infty 1_{\{z \mid x\in B_{z,h}\triangle B_{z,gh}\}}\,dz\,d\mu(x) = \int_0^\infty \mu(B_{z,h}\triangle B_{z,gh})\,dz\,.$$

This way, with h_t as defined in (A.1.2), one deduces from Lemma A.1.2 that as $t \to \infty$,

$$\int_0^\infty \mu(B_{z,h_t})\,dz \to 1 \text{ and } \int_0^\infty \mu(B_{z,h_t}\triangle B_{z,gh_t})\,dz \to 0 \text{ uniformly on compacta.} \tag{A.1.3}$$

Furthermore, uniformly for all $z > 0$ one has

$$\mu(B_{z,h_t}) = \mu\big(\{x \mid h_t(x) \geq z\}\big) \leq \mu\big(\{x \bigm| |\varphi_t(x)| \geq 1\}\big) = \sigma_t\big(\mathbb{R} \smallsetminus (-1,1)\big) \to 0\,, \tag{A.1.4}$$

since by assumption the limit measure is concentrated at 0.

A.1.4. The final step is to get rid of integration over z in (A.1.3). Choose a sequence $\{K_t \mid t \in \mathbb{N}\}$ of compact subsets of G such that:

(i) $e \in K_t$ for all t;
(ii) $K_t \subset K_{t+1}$ for all t, and $\cup_{t=1}^\infty K_t = G$;
(iii) each K_t is equal to the closure of its interior.

Fix a right-invariant Haar measure ν on G. From (i) and (iii) it follows that for any t the value of $\inf_{g\in K_t} \frac{\nu(K_t\cap K_t g)}{\nu(K_t)}$ is positive. Thus one can choose a sequence of positive numbers ε_t with $\varepsilon_t \to 0$ as $t \to \infty$ such that

$$\nu(K_t \cap K_t g) \geq \varepsilon_t \nu(K_t) \text{ for all } g \in K_t\,. \tag{A.1.5}$$

Now, replacing $\{h_t\}$ by a subsequence if needed, in view of (A.1.3) we can assume that for all $g \in K_t$

$$\int_0^\infty \mu(B_{z,h_t} \triangle B_{z,gh_t})\, dz < \frac{\varepsilon_t^2}{4} \int_0^\infty \mu(B_{z,h_t})\, dz\,.$$

Integrating over K_t and then changing the order of integration between dz and $d\nu$, we find that

$$\int_0^\infty \int_{K_t} \left(\tfrac{\varepsilon_t^2}{4} \mu(B_{z,h_t}) - \mu(B_{z,h_t} \triangle B_{z,gh_t}) \right) d\nu(g)\, dz > 0\,.$$

Therefore for every t there exists $z_t > 0$ such that

$$\text{(A.1.6)} \qquad \frac{1}{\nu(K_t)} \int_{K_t} \mu(B_{z_t,h_t} \triangle B_{z_t,gh_t})\, d\nu(g) < \frac{\varepsilon_t^2}{4} \mu(B_{z_t,h_t})\,.$$

Let us now show that the sets $A_t \overset{\text{def}}{=} B_{z_t,h_t}$ form an asymptotically invariant 0-sequence. It is immediate from (A.1.4) that $\mu(B_{z_t,h_t}) \to 0$ as $t \to \infty$. Thus it suffices to find a sequence of compacta $\{K_t'\}$ exhausting G such that

$$\text{(A.1.7)} \qquad \mu(A_t \triangle g A_t)/\mu(A_t) \le \varepsilon_t \text{ whenever } g \in K_t'\,.$$

This is achieved by putting $K_t' \overset{\text{def}}{=} Q_t^{-1} Q_t$, where

$$\text{(A.1.8)} \qquad Q_t \overset{\text{def}}{=} \{g \in K_t \mid \mu(A_t \triangle g A_t) \le \frac{\varepsilon_t}{2} \mu(A_t)\}\,.$$

(Indeed, if $g = g_1^{-1} g_2$, with $g_1, g_2 \in Q_t$, then $\mu(A_t \triangle g A_t) = \mu(g_1 A_t \triangle g_2 A_t) \le \mu(A_t \triangle g_1 A_t) + \mu(A_t \triangle g_2 A_t)$, and (A.1.7) follows.) Therefore, the claim for Case 1 can be derived from condition (ii) and the following

Lemma. *K_t' contains K_t.*

Proof. If not, then there exists $g \in K_t$ such that $Q_t g \cap Q_t = \varnothing$, which implies that $Q_t g \subset (K_t \smallsetminus Q_t) \cup (K_t g \smallsetminus K_t) \quad \Rightarrow \quad \nu(Q_t) \le \nu(K_t) - \nu(Q_t) + \nu(K_t) - \nu(K_t \cap K_t g)$. This, in view of (A.1.5), forces $\nu(Q_t)$ to be not greater than $(1 - \frac{\varepsilon_t}{2})\nu(K_t)$. On the other hand, using (A.1.8) and (A.1.6), one can write

$$\frac{\varepsilon_t}{2} \mu(A_t) \nu(K_t \smallsetminus Q_t) < \int_{K_t \smallsetminus Q_t} \mu(A_t \triangle g A_t)\, d\nu(g) \le \int_{K_t} \mu(A_t \triangle g A_t)\, d\nu(g) < \frac{\varepsilon_t^2}{4} \mu(A_t) \nu(K_t)\,,$$

therefore $\nu(K_t \smallsetminus Q_t) < \frac{\varepsilon_t}{2} \nu(K_t)$, a contradiction. $\square$

Case 2. *The limit measure σ is not concentrated on one point.*

A.2.1. The above assumption implies that there exists $a \in \mathbb{R}$ such that

$$\text{(A.2.1)} \qquad 0 < \sigma\big((a,\infty)\big) = \sigma\big([a,\infty)\big) = \tau < 1\,.$$

Without loss of generality we can assume that $a > 0$. As a first attempt to build a good sequence of sets out of $\{\varphi_t\}$, we consider $B_t \overset{\text{def}}{=} \varphi_t^{-1}\big((a,\infty)\big)$. Then clearly $\mu(B_t) \to \tau$ as $t \to \infty$. Moreover, one has

Lemma. *The sequence $\{B_t\}$ is asymptotically invariant.*

Proof. In view of (A.2.1), for any $\varepsilon > 0$ one can find $\delta > 0$ such that $\sigma\big((a-\delta, a+\delta)\big) = \mu\big(\{x \mid |\varphi_t(x) - a| < \delta\}\big) \le \varepsilon$. Then $\mu(B_t \triangle gB_t)$ is not greater than

$$\mu\big(\{x \mid |\varphi_t(x) - a| < \delta\}\big) + \mu\big(\{x \mid |\varphi_t(x) - a| \ge \delta\} \cap (B_t \triangle gB_t)\big) \le \varepsilon + \frac{1}{\delta^2}\int_X |g\varphi_t - \varphi_t|^2 \, d\mu \, .$$

Since $\{\varphi_t\}$ is asymptotically ρ_0-invariant, $\limsup_{t\to\infty} \mu(B_t \triangle gB_t) \le \varepsilon$ uniformly on compacta, and (AI) follows. □

A.2.2. We now use $\{B_t\}$ to produce a family of asymptotically invariant sequences $B_t^{(k)}$ with $\limsup_{t\to\infty} \mu(B_t^{(k)}) \to 0$ as $k \to \infty$. As a first step, choose a sequence $l_t \to \infty$ and a sequence of increasing compact subsets K_t of G exhausting G such that

(i) $\mu(B_{l_t}) \le \tau + 1/t$;
(ii) $\mu(B_l \triangle gB_l) \le 1/2t$ uniformly in $g \in K_t$ for all $l \ge l_t$;
(iii) $\#\{l \mid \mu(B_{l_t} \cap B_l) > 0\} = \infty$.

Observe that by the Schwarz inequality, for any l one has

$$\left|\int_{B_{l_t} \smallsetminus B_l} \varphi_l \, d\mu\right| \le \left(\int_{B_{l_t} \smallsetminus B_l} \varphi_l^2 \, d\mu\right)^{1/2} \left(\int_{B_{l_t} \smallsetminus B_l} 1 \, d\mu\right)^{1/2} \le \sqrt{\mu(B_{l_t} \smallsetminus B_l)} \, .$$

Therefore

$$\left|\int_{B_{l_t}} \varphi_l \, d\mu\right| \ge \int_{B_{l_t} \cap B_l} \varphi_l \, d\mu - \left|\int_{B_{l_t} \smallsetminus B_l} \varphi_l \, d\mu\right| \ge a\mu(B_{l_t} \cap B_l) - \sqrt{\mu(B_{l_t} \smallsetminus B_l)} \, . \tag{A.2.2}$$

Applying (iii) and the weak convergence of $\{\varphi_t\}$ to zero, for each t choose $l > l_t$ such that $z \overset{\text{def}}{=} \mu(B_{l_t} \cap B_l) > 0$ and $\left|\int_{B_{l_t}} \varphi_l \, d\mu\right| < \frac{a}{2}\mu(B_{l_t})$. Combining this with (A.2.2), we obtain the inequality $az - \sqrt{\mu(B_{l_t}) - z} < \frac{a}{2}\mu(B_{l_t})$. An exercise in quadratic equations gives that z must be less than $\frac{\mu(B_{l_t})}{2} + \frac{\sqrt{1+2a^2\mu(B_{l_t})}-1}{2a^2}$.

Now denote $B_t^{(2)} \overset{\text{def}}{=} B_{l_t} \cap B_l$. Then $\limsup_{t\to\infty} \mu(B_t^{(2)}) \le \tau^{(2)} \overset{\text{def}}{=} \frac{\tau}{2} + \frac{\sqrt{1+2a^2\tau}-1}{2a^2}$. Also, from (ii) it follows that $\mu(B_l \triangle gB_l) \le 1/2t$ and $\mu(B_{l_t} \triangle gB_{l_t}) \le 1/2t$ uniformly in $g \in K_t$. Therefore $\mu(B_t^{(2)} \triangle gB_t^{(2)}) \le \mu(B_l \triangle gB_l) + \mu(B_{l_t} \triangle gB_{l_t}) \le 1/t$, which shows that $\{B_t^{(2)}\}$ is asymptotically invariant.

Applying the above procedure to $\{B_t^{(2)}\}$ we produce another sequence $B_t^{(3)} \overset{\text{def}}{=} B_{l_t}^{(2)} \cap B_l$ for appropriate l_t and $l > l_t$, and, inductively, a family of asymptotically invariant sequences $B_t^{(k)} \overset{\text{def}}{=} B_{l_t}^{(k-1)} \cap B_l$, with $\limsup_{t\to\infty} \mu(B_t^{(k)}) \le \tau^{(k)} \overset{\text{def}}{=} \frac{\tau^{(k-1)}}{2} + \frac{\sqrt{1+2a^2\tau^{(k-1)}}-1}{2a^2}$. It is easy to see that $\tau^{(k)} \to 0$ as $k \to \infty$. Finally, define A_t diagonally as $B_{t'}^{(t)}$, where $t' > t$ is chosen so that $\mu(A_t \triangle gA_t) < 1/t$ whenever g belongs to the compact set K_t. This completes the construction of the asymptotically invariant 0-sequence $\{A_t\}$, as well as the proof of Proposition A.0. □

A.3. It remains to write down the

Proof of Lemma 3.1. It is easy to deduce from (3.1) and the G-equivariance of π that if $\{A_t\}$ is an asymptotically invariant 0-sequence of subsets of X_1, then $\{\pi(A_t)\}$ is an asymptotically invariant 0-sequence of subsets of X_2; and, conversely, any asymptotically invariant 0-sequence $\{A_t\}$ of subsets of X_2 gives rise to an asymptotically invariant 0-sequence $\{\pi^{-1}(A_t)\}$ of subsets of X_1. □

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References

[B] M. Bekka, *On uniqueness of invariant means*, Proc. Amer. Math. Soc. **126** (1998), 507–514.

[Bo] A. Borel, *Introduction aux groupes arithmétiques*, Hermann, Paris, 1969.

[Bos] M. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.

[Bou1] N. Bourbaki, *Integration, Chapitres VII et VIII*, Hermann, Paris, 1963.

[Bou2] ———, *Groupes et Algèbres de Lie, Chapitres IV, V et VI*, Hermann, Paris, 1968.

[BS] M. Burger and P. Sarnak, *Ramanujan Duals II*, Inv. Math. **106** (1991), 1–11.

[CK] N. Chernov and D. Kleinbock, *Dynamical Borel-Cantelli lemmas for Gibbs measures*, Preprint (1999).

[CR] J.-P. Conze and A. Raugi, *Convergence des potentiels pour un opérateur de transfert, applications aux systèmes dynamiques et aux chaînes de Markov*, Preprint (1999).

[D] S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. Reine Angew. Math. **359** (1985), 55–89.

[Do] M. M. Dodson, *Geometric and probabilistic ideas in metric Diophantine approximation*, Russian Math. Surveys **48** (1993), 73–102.

[Dol] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Preprint (1999).

[F] Y. Flicker, *Automorphic forms on covering groups of $GL(2)$*, Inv. Math. **57** (1980), 119–182.

[FS] A. Furman and Y. Shalom, *Sharp ergodic theorems for group actions and strong ergodicity*, Erg. Th. Dyn. Systems (to appear).

[G] A. V. Groshev, *Une théorème sur les systèmes des formes linéaires*, Dokl. Akad. Nauk SSSR **9** (1938), 151–152.

[GR] H. Garland and M. S. Raghunathan, *Fundamental domains for lattices in R-rank* 1 *semisimple Lie groups*, Ann. Math. **92** (1970), 279–326.

[HV] R. Hill and S. Velani, *Ergodic theory of shrinking targets*, Invent. Math. **119** (1995), 175–198.

[J] L. Ji, *Metric compactifications of locally symmetric spaces*, Internat. J. Math. **9** (1998), 465–491.

[KM] D. Kleinbock and G. A. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Amer. Math. Soc. Transl. **171** (1996), 141–172.

[KS] S. Kochen and C. Stone, *A note on the Borel Cantelli lemma*, Ill. J. Math. **8** (1964), 248–251.

[L] E. Leuzinger, *Tits geometry and arithmetic groups*, Preprint (1997).

[M] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin and New York, 1991.

[Ma] F. J. Mautner, *Geodesic flows on symmetric Riemannian spaces*, Ann. Math. **65** (1957), 416–431.

[N] I. Namioka, *Folner's conditions for amenable semi-groups*, Math. Scand. **15** (1964), 18–28.

[P] W. Philipp, *Some metrical theorems in number theory*, Pacific J. Math. **20** (1967), 109–127.

[R1] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, Berlin and New York, 1972.

[R2] ———, *On the congruence subgroup problem*, Publ. Math. IHES **46** (1976), 107–161.

[Ro] J. Rosenblatt, *Uniqueness of invariant means for measure preserving transformations*, Trans. Amer. Math. Soc. **265** (1981), 623–636.

[S] K. Schmidt, *Amenability, Kazhdan's property (T), strong ergodicity and invariant means for ergodic group actions*, Erg. Th. Dyn. Systems **1** (1981), 223–236.

[Si] C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. Math. **46** (1945), 340–347.

[Sk] M. Skriganov, *Ergodic theory on $SL(n)$, Diophantine approximations and anomalies in the lattice point problem*, Invent. Math. **132** (1998), 1–72.

[Sp] F. Spitzer, *Principles of random walk*, Van Nostrand, Princeton, 1964.

[Spr] V. Sprindžuk, *Metric theory of Diophantine approximations*, John Wiley & Sons, New York-Toronto-London, 1979.

[Su] D. Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982), 215–237.

[T1] J. Tits, *Algebraic and abstract simple groups*, Ann. Math. **80** (1964), 313–329.

[T2] ———, *Classification of algebraic semisimple groups*, 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 33–62.

[V] M.-F. Vigneras, *Quelques remarques sur la conjecture* $\lambda_1 \geq \frac{1}{4}$, Seminar in number theory, Paris 1981-82, Progr. Math., 38, Birkhäuser Boston, Boston, MA, 1983, pp. 321–343.

[W] G. Warner, *Harmonic analysis on semisimple Lie groups I*, Springer-Verlag, Berlin and New York, 1972.

[Z] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.

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